

Two or more random variables

Two continuous random variables

If X and Y are continuous random variables defined on the same sample space, they have a *joint probability density function* $f_{X,Y}(x,y)$ such that, if A is any region of $\mathbb{R}^2$, then

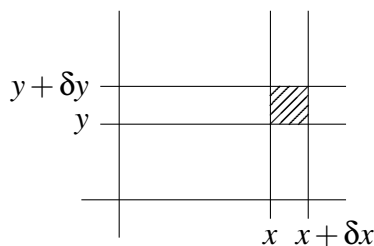
$$\mathbb{P}((X,Y) \in A) = \int \int_{(x,y) \in A} f_{X,Y}(x,y) dx dy.$$

Since probabilities are non-negative, we have $f_{X,Y}(x,y) \geq 0$ for all real x and y . Putting $A = \mathbb{R}^2$ gives

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1.$$

It can also be shown that if δx and δy are small and positive then

$$\mathbb{P}(x \leq X \leq x + \delta x \text{ and } y \leq Y \leq y + \delta y) \approx f_{X,Y}(x,y) \delta x \delta y.$$



The *joint cumulative distribution function* $F_{X,Y}$ is given by

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x \text{ and } Y \leq y) = \int_{-\infty}^y \int_{-\infty}^x f_{X,Y}(t,u) dt du.$$

The *marginal probability density functions* are obtained by integrating over the other variable:

$$\begin{aligned}f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy; \\f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx.\end{aligned}$$

If g is a real function of two variables and if X and Y have a joint continuous distribution then $\mathbb{E}(g(X,Y))$ is defined by

$$\mathbb{E}(g(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) \, dx \, dy.$$

Theorem 8 $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.

The proof is just like the proof in the discrete case, but uses integration instead of summation.

For continuous random variables, $\mathbb{P}(X=x) = \mathbb{P}(Y=y) = 0$, so we have to adapt the definition of independence.

Definition If X and Y have a joint continuous distribution then X and Y are *independent* of each other if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all real x and y .

If X and Y are independent of each other and $x_1 < x_2$ and $y_1 < y_2$ then

$$\begin{aligned}\mathbb{P}(x_1 \leq X \leq x_2 \text{ and } y_1 \leq Y \leq y_2) &= \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{X,Y}(x,y) \, dx \, dy \\&= \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_X(x) f_Y(y) \, dx \, dy \\&= \int_{y_1}^{y_2} f_Y(y) \left[\int_{x_1}^{x_2} f_X(x) \, dx \right] \, dy \\&= \int_{y_1}^{y_2} f_Y(y) \mathbb{P}(x_1 \leq X \leq x_2) \, dy \\&= \mathbb{P}(x_1 \leq X \leq x_2) \int_{y_1}^{y_2} f_Y(y) \, dy \\&= \mathbb{P}(x_1 \leq X \leq x_2) \mathbb{P}(y_1 \leq Y \leq y_2)\end{aligned}$$

so the events “ $x_1 \leq X \leq x_2$ ” and “ $y_1 \leq Y \leq y_2$ ” are independent of each other.

The following holds for joint continuous random variables as well as for joint discrete random variables. Again, the proof uses integration instead of summation.

Theorem 9 If X and Y are independent then $\mathbb{E}(XY) = \mathbb{E}(X) \mathbb{E}(Y)$.

Covariance and correlation

Definition The *covariance* of X and Y is

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))].$$

This definition is exactly the same as it was for discrete random variables. So are the following two theorems and their proofs.

Theorem 10 (a) $\text{Cov}(X, X) = \text{Var}(X)$.

(b) $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$.

(c) If a is a constant then $\text{Cov}(aX, Y) = \text{Cov}(X, aY) = a\text{Cov}(X, Y)$.

(d) If b is a constant then $\text{Cov}(X, Y + b) = \text{Cov}(X + b, Y) = \text{Cov}(X, Y)$.

(e) $\text{Cov}(X, Y + Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z)$.

(f) If X and Y are independent then $\text{Cov}(X, Y) = 0$.

Theorem 11 If a and b are constants and X and Y are random variables then

$$\text{Var}(aX + bY) = a^2 \text{Var}(X) + 2ab \text{Cov}(X, Y) + b^2 \text{Var}(Y).$$

In particular, if X and Y are independent of each other then

$$\text{Var}(X + Y) = \text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y).$$

Definition The *correlation* between random variables X and Y is

$$\text{corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}.$$

Theorem 12 $-1 \leq \text{corr}(X, Y) \leq 1$.

Proof Put $\mathbb{E}(X) = \mu_X$ and $\mathbb{E}(Y) = \mu_Y$. Let t be any real number. Then

$$[(X - \mu_X) + t(Y - \mu_Y)]^2 \geq 0,$$

so

$$\mathbb{E} \left([(X - \mu_X) + t(Y - \mu_Y)]^2 \right) \geq 0.$$

$$\begin{aligned}
\text{The left-hand side} &= \mathbb{E}[(X - \mu_X)^2 + 2t(X - \mu_X)(Y - \mu_Y) + t^2(Y - \mu_Y)^2] \\
&= \mathbb{E}[(X - \mu_X)^2] + 2t \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] + t^2 \mathbb{E}[(Y - \mu_Y)^2] \\
&= \text{Var}(X) + 2t \text{Cov}(X, Y) + t^2 \text{Var}(Y) \\
&= f(t), \quad \text{say.}
\end{aligned}$$

Then $f(t)$ is a quadratic function of t which is always non-negative, so its discriminant (the part we think of as “ $b^2 - 4ac$ ”) is negative or zero: that is,

$$4[\text{Cov}(X, Y)]^2 - 4 \text{Var}(X) \text{Var}(Y) \leq 0.$$

The quantity $4 \text{Var}(X) \text{Var}(Y)$ is positive, so we can divide by it without changing the sign, so

$$\frac{[\text{Cov}(X, Y)]^2}{\text{Var}(X) \text{Var}(Y)} - 1 \leq 0,$$

so

$$\frac{[\text{Cov}(X, Y)]^2}{\text{Var}(X) \text{Var}(Y)} \leq 1.$$

That is,

$$[\text{corr}(X, Y)]^2 \leq 1.$$

Taking square roots, we obtain

$$-1 \leq \text{corr}(X, Y) \leq 1. \quad \blacksquare$$

Theorem 13 Let X and Y be random variables such that $Y = aX + b$, where a and b are constants and $a \neq 0$. Then

$$\text{corr}(X, Y) = \begin{cases} +1 & \text{if } a \text{ is positive} \\ -1 & \text{if } a \text{ is negative.} \end{cases}$$

Proof By Theorem 10, $\text{Cov}(X, Y) = \text{Cov}(X, aX + b) = \text{Cov}(X, aX) = a \text{Cov}(X, X) = a \text{Var}(X)$. Also $\text{Var}(Y) = \text{Var}(aX + b) = \text{Var}(aX) = a^2 \text{Var}(X)$, and so $\text{Var}(X) \text{Var}(Y) = a^2 (\text{Var}(X))^2$.

If $a > 0$ then the positive square root of $a^2 (\text{Var}(X))^2$ is $a \text{Var}(X)$, so

$$\text{corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{a \text{Var}(X)}{a \text{Var}(X)} = 1.$$

If $a < 0$ then the positive square root of $a^2 (\text{Var}(X))^2$ is $-a \text{Var}(X)$, so

$$\text{corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{a \text{Var}(X)}{-a \text{Var}(X)} = -1. \quad \blacksquare$$