3.5 Efficiency factors

For comparison we consider a complete-block design where the variance of each response is σ^2_{CBD}. In such a design, $\Lambda = rJ_\Theta$ and $k = t$, so Equation (3.3) gives

$$L = r(I_\Theta - t^{-1}J_\Theta).$$

Now, $(I_\Theta - t^{-1}J_\Theta)$ is the projector onto $U_0^\perp$, so

$$L^- = \frac{1}{r}(I_\Theta - t^{-1}J_\Theta)$$

and the variance of the estimator of $x'\tau$ is $(x'L^-x)\sigma^2_{\text{CBD}}$, which is equal to $r^{-1}x'x\sigma^2_{\text{CBD}}$.

Definition The *efficiency* for a contrast x in an equi-replicate incomplete-block design with variance σ^2 and replication r relative to a complete-block design with variance σ^2_{CBD} and the same replication is

$$\frac{x'x}{rx'L^-x}\frac{\sigma^2_{\text{CBD}}}{\sigma^2}$$

and the *efficiency factor* for x is

$$\frac{x'x}{rx'L^-x}.$$

If x is the simple contrast for the difference between treatments θ and η then $x'x = 2$. Thus

$$\text{Var}(\widehat{\tau(\theta) - \tau(\eta)}) = \frac{2}{r}\frac{\sigma^2}{\text{efficiency factor for } x}. \qquad (3.4)$$

Example 3.1 revisited Here $r = 2$, so the efficiency factor for a simple contrast is $\sigma^2/\text{variance}$. Thus the efficiency factor for the simple contrast $\chi_1 - \chi_2$ is 1 while the efficiency factor for the simple contrast $\chi_1 - \chi_3$ is $6/7$.

When $x = \chi_1 + \chi_2 - \chi_3 - \chi_4$ then $x'x = 4$ and $\text{Var}(\widehat{x'\tau}) = 8\sigma^2/3$, so the efficiency factor for x is

$$\frac{4}{2} \times \frac{3}{8} = \frac{3}{4}. \quad \blacksquare$$

We want estimators with low variance. Efficiency is defined by comparing the reciprocals of the variances, so that low variance corresponds to high efficiency. In practice, neither σ^2 nor σ^2_{CBD} is known before the experiment is done. Indeed, the usual reason for doing an experiment in incomplete blocks is that large enough blocks are not available. Even if they are available, there may be some prior knowledge about the likely relative sizes of σ^2 and σ^2_{CBD}. Part of the statistician's job in deciding what blocks to use is to assess whether the ratio $\sigma^2/\sigma^2_{\mathrm{CBD}}$ is likely to be less than the ratio $x'x/rx'L^-x$. The latter ratio, the efficiency factor, is a function of the design and the contrast, so it can be used for comparing different designs of the same size, at least for the single contrast x.

The efficiency factor for x has a particularly simple form if x is an eigenvector of L, for if $Lx = \mu x$ then $x'L^-x = \mu^{-1}x'x$ and so the efficiency factor is μ/r.

Definition A *basic contrast* of an equi-replicate incomplete-block design is a contrast which is an eigenvector of the information matrix L.

Definition The *canonical efficiency factors* of an equi-replicate incomplete-block design with replication r are $\mu_1/r, \ldots, \mu_{t-1}/r$, where $\mu_1, \ldots, \mu_{t-1}$ are the eigenvalues of L on $U_0^\perp$, with multiplicities.

Technique 3.1 To find the canonical efficiency factors, find the eigenvalues of Λ, divide by rk, subtract from 1, and ignore one of the zero values. There will always be a zero value corresponding to the eigenvector χ_Θ; if the design is connected then that will be the only zero value.

Once the eigenvectors and eigenvalues of L are known, they can be used to find the efficiency factors of all contrasts if the design is connected.

Technique 3.2 If the eigenvalues of L are known, use them to write down the minimal polynomial of L on $(\ker L)^\perp$. Hence find the Moore-Penrose generalized inverse L^- of L. If the design is connected and x is any contrast then $x'L^-x$ is not zero, so calculate the efficiency factor for x as $x'x/rx'L^-x$. Ignore the contribution of J to L^-, because $Jx = 0$ for all contrasts x.

Example 3.1 revisited We have seen that $Lx = 2x$ if x is $\chi_1 - \chi_2$, $\chi_3 - \chi_4$ or $\chi_5 - \chi_6$ and $Lx = (3/2)x$ if x is $\chi_1 + \chi_2 - \chi_3 - \chi_4$ or $\chi_1 + \chi_2 - \chi_5 - \chi_6$. These five contrasts span $U_0^\perp$, so the eigenvalues of L on $U_0^\perp$ are 2 and $3/2$. Thus, on $U_0^\perp$,

$$(L - 2I)(L - \frac{3}{2}I) = 0,$$

whence

$$L^2 - \frac{7}{2}L + 3I = 0$$

so

$$L(L - \frac{7}{2}I) = -3I$$

and the inverse of L on $U_0^\perp$ is $(1/6)(7I-2L)$. Thus

$$L^- = \frac{1}{6}(7I-2L)+cJ$$

for some constant c. If x is a contrast then $Jx=0$ and so

$$x'L^-x = \frac{1}{6}(7x'x-2x'Lx).$$

In particular, if $x=\chi_1-\chi_3$ then $x'x=2$ and

$$x'L^-x = \frac{1}{6}\left(14-2\left(L(1,1)-L(1,3)-L(3,1)+L(3,3)\right)\right) = \frac{7}{6},$$

so the efficiency factor for $\chi_1-\chi_3$ is $6/7$, as we found on page 63. ■

Technique 3.3 If the basic contrasts and their canonical efficiency factors are known, express an arbitrary contrast x as a sum $x_1+\cdots+x_s$ of basic contrasts with different canonical efficiency factors $\varepsilon_1, \ldots, \varepsilon_s$. If the design is connected then none of $\varepsilon_1, \ldots, \varepsilon_s$ is zero. Since the distinct eigenspaces of L are orthogonal to each other, $x_i'L^-x_j=0$ if $i\neq j$, so

$$rx'L^-x = \sum_{i=1}^{s}\frac{x_i'x_i}{\varepsilon_i}.$$

Example 3.1 revisited Put $x=\chi_1-\chi_3$. Then $x=x_1+x_2$ where $x_1=(\chi_1-\chi_2-\chi_3+\chi_4)/2$ and $x_2=(\chi_1+\chi_2-\chi_3-\chi_4)/2$. But x_1 is a basic contrast with canonical efficiency factor $\varepsilon_1=1$ and x_2 is a basic contrast with canonical efficiency factor $\varepsilon_2=3/4$, so

$$rx'L^-x = x_1'x_1+\frac{4}{3}x_2'x_2 = 1+\frac{4}{3}=\frac{7}{3}$$

and $x'x/rx'L^-x=6/7$, as before. ■

We need an overall measure of the efficiency of a design. It is tempting to take the arithmetic mean of the canonical efficiency factors. But

$$\begin{aligned}\sum \text{canonical efficiency factors} &= \sum \text{canonical efficiency factors}+0\\ &= \frac{1}{r}\left(\sum \text{eigenvalues of } L\right)\\ &= \frac{1}{r}\operatorname{tr}L = \frac{1}{r}\left(rt-\frac{rt}{k}\right)\\ &= \frac{t(k-1)}{k}, \qquad (3.5)\end{aligned}$$

which is independent of the design. Instead we measure the overall efficiency of the design by the harmonic mean of the canonical efficiency factors. (The harmonic mean of a collection of positive numbers is the reciprocal of the arithmetic mean of their reciprocals.) This overall efficiency factor is called A (not to be confused with an adjacency matrix!). That is, if the canonical efficiency factors are $\varepsilon_1, \ldots, \varepsilon_{t-1}$ then

$$A = \left(\frac{\sum_{i=1}^{t-1} \frac{1}{\varepsilon_i}}{t-1} \right)^{-1}.$$

The next theorem shows that the choice of harmonic mean is not entirely arbitrary.

Theorem 3.8 *In a connected equi-replicate incomplete-block design with replication r, the average variance of simple contrasts is equal to* $2\sigma^2/(rA)$.

Proof The information matrix is zero in its action on U_0, so the same is true of its generalized inverse L^-. That is, the row and column sums of L^- are all zero. Equation (3.2) shows that the average variance of simple contrasts

$$\begin{aligned}
&= \frac{\sigma^2}{t(t-1)} \sum_\eta \sum_{\theta \neq \eta} \left(L^-(\theta,\theta) - L^-(\theta,\eta) - L^-(\eta,\theta) + L^-(\eta,\eta) \right) \\
&= \frac{\sigma^2}{t(t-1)} \sum_\eta \sum_{\theta} \left(L^-(\theta,\theta) - L^-(\theta,\eta) - L^-(\eta,\theta) + L^-(\eta,\eta) \right) \\
&= \frac{\sigma^2}{t(t-1)} \sum_\eta \sum_{\theta} \left(L^-(\theta,\theta) + L^-(\eta,\eta) \right) \qquad \text{because the row and column sums of } L^- \text{ are zero} \\
&= \frac{\sigma^2}{t(t-1)} 2t \operatorname{tr} L^- \\
&= \frac{2\sigma^2}{t-1} \left(\frac{1}{\mu_1} + \cdots + \frac{1}{\mu_{t-1}} \right) \qquad \text{where } \mu_1, \ldots, \mu_{t-1} \text{ are the eigenvalues of } L \text{ on } U_0^\perp \\
&= \frac{2\sigma^2}{r(t-1)} \left(\frac{r}{\mu_1} + \cdots + \frac{r}{\mu_{t-1}} \right) \\
&= \frac{2\sigma^2}{r} \times \frac{1}{\text{harmonic mean of } \frac{\mu_1}{r}, \ldots, \frac{\mu_{t-1}}{r}} \\
&= \frac{2\sigma^2}{rA}. \quad \blacksquare
\end{aligned}$$

Theorem 3.9 *The canonical efficiency factors of an equi-replicate incomplete-block design and its dual are the same, including multiplicities, apart from* $|b-t|$ *values equal to* 1.

Proof Let ε be a canonical efficiency factor different from 1 for the original design and let x be a corresponding eigenvector of L. Then $Lx = r\varepsilon x$. From Lemma 3.1 and Equation (3.3), $N'Nx = \Lambda x = k(rI - L)x = kr(1-\varepsilon)x$. Therefore $NN'Nx = kr(1-\varepsilon)Nx$. The dual design has replication k and information matrix $kI_\Delta - r^{-1}NN'$. Now,

$$\left(kI_\Delta - \frac{1}{r}NN'\right)Nx = k\varepsilon Nx.$$

Thus x has canonical efficiency factor equal to ε in the original design and Nx has canonical efficiency factor equal to ε in the dual.

The maps

$$x \mapsto Nx \qquad \text{and} \qquad y \mapsto \frac{1}{rk(1-\varepsilon)}N'y$$

are mutual inverses on the spaces of contrasts in the two designs which have canonical efficiency factor ε, so the dimensions of these spaces are equal.

All remaining canonical efficiency factors of both designs must be equal to 1. ■

Note that if the canonical efficiency factors are $\varepsilon_1, \ldots, \varepsilon_{t-1}$ then the $rk\varepsilon_i$ are the zeros of the monic integer polynomial

$$\det(\mathrm{x}I - kL) \tag{3.6}$$

in $\mathbb{Z}[\mathrm{x}]$. So each $rk\varepsilon_i$ is an algebraic integer, so is either an integer or irrational. This fact helps to identify the canonical efficiency factors exactly if a computer program finds them numerically. Moreover,

$$\frac{1}{rk}\sum \frac{1}{\varepsilon_i} = \frac{1}{rk}\frac{\sum_i \prod_{j\neq i}\varepsilon_j}{\prod_i \varepsilon_i} = \frac{\sum_i \prod_{j\neq i} rk\varepsilon_j}{\prod_i rk\varepsilon_i}:$$

both numerator and denominator are elementary symmetric functions in the zeros of the polynomial (3.6), so they are integers. Hence A is always rational.

Definition An equi-replicate incomplete-block design is *A-optimal* if it has the highest value of A among all incomplete-block designs with the same values of t, r, b and k.

3.6 Variance and efficiency in partially balanced designs

Theorem 3.10 *In a partially balanced incomplete-block design with non-diagonal associate classes* $\mathcal{C}_1, \ldots, \mathcal{C}_s$ *there are constants* $\kappa_1, \ldots, \kappa_s$ *such that the variance of the estimator of* $\tau(\theta) - \tau(\eta)$ *is equal to* $\kappa_i\sigma^2$ *if* $(\theta, \eta) \in \mathcal{C}_i$*.*

Proof By definition, $\Lambda = \sum_{i=0}^{s} \lambda_i A_i \in \mathcal{A}$ so $L \in \mathcal{A}$. From Section 2.2, $L^- \in \mathcal{A}$. Thus there are constants $\nu_0, \ldots, \nu_s$ such that $L^- = \sum_{i=0}^{s} \nu_i A_i$. Now Equation (3.2) shows that

$$\begin{aligned} \operatorname{Var}(\widehat{\tau(\theta) - \tau(\eta)}) &= \left(L^-(\theta,\theta) - L^-(\theta,\eta) - L^-(\eta,\theta) + L^-(\eta,\eta)\right)\sigma^2, \\ &= (\nu_0 - \nu_i - \nu_i + \nu_0)\sigma^2 \qquad \text{if } (\theta,\eta) \in \mathcal{C}_i \\ &= \kappa_i\sigma^2 \end{aligned}$$

with $\kappa_i = 2(\nu_0 - \nu_i)$. ■

Theorem 3.10 shows why partially balanced incomplete-block designs were invented. To a combinatorialist the pattern of concurrences which defines partial balance is interesting. To a statistician, the pattern of variances demonstrated by Theorem 3.10 is important, far more important than combinatorial patterns. Many statisticians are puzzled that in general incomplete-block designs the pattern of variances of simple contrasts does not match the pattern of concurrences. The technical condition about p_{ij}^k in the definition of association scheme is there precisely to give Theorem 3.10. The irony is that many statisticians who are interested in the pattern of variances reject the p_{ij}^k condition as 'too mathematical'.

Example 3.10 In a balanced incomplete-block design, $\Lambda = rI + \lambda(J - I)$, so

$$L = \left(\frac{r(k-1)+\lambda}{k}\right) I - \frac{\lambda}{k} J = \frac{\lambda t}{k}\left(I - \frac{1}{t}J\right),$$

because $\lambda(t-1) = r(k-1)$. Thus

$$L^- = \frac{k}{\lambda t}\left(I - \frac{1}{t}J\right),$$

so

$$\operatorname{Var}(\widehat{\tau(\theta) - \tau(\eta)}) = \frac{k}{\lambda t}(1+1)\sigma^2 = \frac{2k}{\lambda t}\sigma^2 = \frac{2}{r}\frac{k}{(k-1)}\frac{(t-1)}{t}\sigma^2.$$

Equation (3.4) shows that the efficiency factor for every simple contrast is equal to

$$\frac{t}{t-1}\frac{k-1}{k},$$

which is indeed the eigenvalue of $r^{-1}L$ on the whole of $U_0^\perp$. ■

Theorem 3.11 (Fisher's Inequality) *If a balanced incomplete-block design has t treatments and b blocks then $b \geqslant t$.*

Proof If $t > b$ then Theorem 3.9 shows that the design has at least $t - b$ canonical efficiency factors equal to 1. But Example 3.10 shows that no canonical efficiency factor is equal to 1 in a BIBD. ■

Technique 3.4 Pretending that $J = O$, find the inverse of L on $U_0^\perp$. If L is a polynomial in a single adjacency matrix A, use the minimal polynomial of A on $U_0^\perp$ to calculate this inverse.

Example 3.7 revisited Let A be the adjacency matrix for first associates in the triangular association scheme $\mathrm{T}(n)$. The argument at the end of Section 1.4.4, or the parameters given in Section 1.4.1, show that

$$A^2 = (2n-4)I + (n-2)A + 4(J - A - I).$$

The incomplete-block design in Example 3.7 is partially balanced with respect to the triangular association scheme $\mathrm{T}(5)$, for which

$$A^2 = 2I - A + 4J.$$

We have $\Lambda = 3I + A$ and thus $L = \frac{1}{3}(6I - A)$. If we pretend that $J = O$ then

$$LA = \frac{1}{3}(6A - A^2) = \frac{1}{3}(6A - 2I + A) = \frac{1}{3}(7A - 2I)$$

and so

$$L(A + 7I) = \frac{40}{3}I.$$

Therefore

$$L^- = \frac{3}{40}(A + 7I) + cJ$$

for some c. If θ and η are first associates then the variance of the estimator of $\tau(\theta) - \tau(\eta)$ is

$$2\sigma^2 \times \frac{3}{40} \times (7-1) = \frac{9\sigma^2}{10};$$

otherwise it is $21\sigma^2/20$.

Each treatment has six first associates and three second associates, so the average variance of simple contrasts is

$$\frac{1}{3}\left[2 \times \frac{18\sigma^2}{20} + \frac{21\sigma^2}{20}\right] = \frac{19\sigma^2}{20}.$$

By Theorem 3.8, the harmonic mean efficiency factor A is equal to $40/57$. ■

Theorem 3.12 *In a partially balanced incomplete-block design, the strata (in $\mathbb{R}^\Theta$) are sub-eigenspaces of the information matrix, and the canonical efficiency factors are*

$$1 - \frac{1}{rk}\sum_{i=0}^{s} \lambda_i C(i,e)$$

with multiplicity d_e, for e in $\mathcal{E}\setminus\{0\}$, where C is the character table of the association scheme.

Proof We have

$$L = rI - \frac{1}{k}\Lambda = rI - \frac{1}{k}\sum_{i=0}^{s}\lambda_i A_i.$$

If x is in the stratum U_e then $A_i x = C(i,e)x$ and so

$$Lx = rx - \frac{1}{k}\sum_{i=0}^{s}\lambda_i C(i,e)x. \quad \blacksquare$$

Example 3.11 Consider nine treatments in a 3×3 square, forming the Hamming association scheme $\mathrm{H}(2,3)$. The nine blocks of shape

	*	*
*		
*		

give a partially balanced incomplete-block design with $t = b = 9$, $k = 4$, $\lambda_0 = 4$, $\lambda_1 = 1$ and $\lambda_2 = 2$.

Now $A_1^2 = 4I + A_1 + 2A_2 = 4I + A_1 + 2(J - I - A_1)$. Ignoring J, we get $A_1^2 + A_1 - 2I = O$ so $(A_1 + 2I)(A_1 - I) = O$. This leads to the character table

$$\begin{array}{lll} & & \\ \lambda_0 = 4 & \text{0th associates} & (1) \\ \lambda_1 = 1 & \text{1st associates} & (4) \\ \lambda_2 = 2 & \text{2nd associates} & (4) \end{array} \begin{array}{c} \begin{array}{ccc} (1) & (4) & (4) \end{array} \\ \left[\begin{array}{rrr} 1 & 1 & 1 \\ 4 & -2 & 1 \\ 4 & 1 & -2 \end{array}\right] \end{array}$$

We use Theorem 3.12 on the three columns of the character table:

$$\begin{aligned} 1 - \frac{1}{16}(4+4+2\times 4) &= 0, \quad \text{as it must do,} \\ 1 - \frac{1}{16}(4-2+2\times 1) &= \frac{3}{4} \\ 1 - \frac{1}{16}(4+1+2\times(-2)) &= \frac{15}{16}. \end{aligned}$$

Thus the canonical efficiency factors are $3/4$ and $15/16$, with multiplicity 4 each, and

$$A = \left(\frac{\frac{4}{3} + \frac{16}{15}}{2} \right)^{-1} = \frac{5}{6}. \quad \blacksquare$$

Technique 3.5 Even if you do not remember the whole character table for an association scheme, do remember its strata. Find the canonical efficiency factor for each stratum by applying Λ to any vector in that stratum.

Example 3.12 The group-divisible design in Example 3.4 has $k = 3$, $r = 2$ and groups $a, f \parallel b, e \parallel c, d \parallel$. The concurrence matrix is

$$\Lambda = \begin{array}{c} \\ a \\ f \\ b \\ e \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccc} a & f & b & e & c & d \end{array} \\ \left[\begin{array}{cccccc} 2 & 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2 \end{array} \right] \end{array}$$

One within-groups vector is $\chi_a - \chi_f$, which is an eigenvector of Λ with eigenvalue 2. So the within-groups canonical efficiency factor is equal to $1 - 2/6 = 2/3$, with multiplicity 3. One between-groups vector is $\chi_a + \chi_f - \chi_b - \chi_e$. This is an eigenvector of Λ with eigenvalue 0, so the between-groups canonical efficiency factor is equal to 1, with multiplicity 2.

The dual design is balanced, with all canonical efficiency factors equal to $\frac{4}{3} \times \frac{1}{2} = \frac{2}{3}$, as shown in Example 3.10. This agrees with Theorem 3.9. $\blacksquare$

Technique 3.6 If you remember the stratum projectors for the association scheme, express L in terms of them. The coefficients of the projectors are the eigenvalues of L. Moreover, L^- is obtained by replacing each non-zero coefficient by its reciprocal.

Example 3.12 revisited As we saw in Section 2.3, it is useful to let G be the adjacency matrix for the relation "is in the same group as". Then the stratum projectors are

$$\frac{1}{6}J, \quad \frac{1}{2}G - \frac{1}{6}J \quad \text{and} \quad I - \frac{1}{2}G,$$

with corresponding dimensions 1, 2 and 3 respectively. Now, $\Lambda = 2I + (J - G)$ so

$$L = 2I - \frac{1}{3}(2I + J - G) = \frac{4I - J + G}{3} = \frac{4}{3}\left(I - \frac{1}{2}G\right) + 2\left(\frac{1}{2}G - \frac{1}{6}J\right).$$

(Note that the coefficient of $t^{-1}J$ *must* be zero, so here is a check on the arithmetic.) From this we read off the eigenvalues of L as $4/3$ and 2, with multiplicities 3 and 2 respectively. Dividing these by 2 gives the canonical efficiency factors $2/3$ and 1 that we found before.

Moreover,

$$L^{-} = \frac{3}{4}\left(I - \frac{1}{2}G\right) + \frac{1}{2}\left(\frac{1}{2}G - \frac{1}{6}J\right).$$

For treatments that are first associates, the relevant 2×2 submatrices of G and J are both equal to

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

which make no contribution to the variance of the difference, which is therefore equal to $2 \times (3/4)\sigma^2 = (3/2)\sigma^2$. This agrees with what we already know, because we have already found that the efficiency factor for the simple contrast of first associates is equal to $2/3$. From Equation (3.4), the variance is equal to

$$\frac{2}{r}\frac{\sigma^2}{\text{efficiency factor}},$$

which is equal to

$$\frac{2}{2}\frac{3\sigma^2}{2}$$

in this case.

For treatments that are second associates we can still ignore J, but the relevant 2×2 submatrices of G and I are both equal to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so the variance of the difference is equal to

$$2\left[\frac{3}{4}\left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right)\right]\sigma^2 = \frac{5}{4}\sigma^2.$$

Each treatment has one first associate and four second associates, so the average variance of simple contrasts is

$$\frac{\frac{3}{2} + 4 \times \frac{5}{4}}{5}\sigma^2 = \frac{13}{10}\sigma^2.$$

We can also calculate A directly as

$$A = \left(\frac{3 \times \frac{3}{2} + 2 \times 1}{5}\right)^{-1} = \frac{10}{13}.$$

The values of A and of the average variance are in agreement with Theorem 3.8. ■

Technique 3.7 Calculate the canonical efficiency factors and then check the arithmetic by verifying that Equation (3.5) holds. Alternatively, if there is one stratum whose vectors are more difficult for calculations, then find the other canonical efficiency factors and deduce the missing canonical efficiency factor from Equation (3.5); that is

$$\sum_{e=1}^{s} d_e \varepsilon_e = \frac{t(k-1)}{k},$$

where ε_e is the canonical efficiency factor for stratum U_e.