3.2 Partially balanced incomplete-block designs

Definition An incomplete-block design with treatment-set Θ is *partially balanced* with respect to a given association scheme on Θ with non-diagonal associate classes $\mathcal{C}_1$, $\mathcal{C}_2$, ..., $\mathcal{C}_s$ if there are integers λ_0, λ_1, ..., λ_s such that

$$(\theta,\eta) \in \mathcal{C}_i \Longrightarrow \Lambda(\theta,\eta) = \lambda_i;$$

in other words

$$\Lambda = \sum_{i=0}^{s} \lambda_i A_i.$$

More generally, an incomplete-block design is partially balanced if there exists some association scheme on the treatment set with respect to which it is partially balanced.

Partial balance implies equal replication. We always write r for λ_0.

Definition An incomplete-block design is *balanced* if it is partially balanced with respect to the trivial association scheme.

Statisticians often abbreviate partially balanced incomplete-block designs to PBIBDs and balanced incomplete-block designs to BIBDs. On the other hand, pure mathematicians often know BIBDs as '2-designs'; some people call them simply 'designs'. We always write λ for λ_1 in a BIBD.

Of course, a given incomplete-block design may be partially balanced with respect to more than one association scheme on Θ. For example, a BIBD is partially balanced with respect to every association scheme on Θ. We usually give the association scheme with the smallest number of associate classes among those with respect to which the design is partially balanced. In Chapter **??** we shall show that such a 'simplest' association scheme is well defined.

Definition An incomplete-block design is a *group divisible*, *triangular*, ... design if it is partially balanced with respect to any group divisible, triangular, ... association scheme on Θ. It is a *transversal design* if it is group divisible with $\lambda_1 = 0$ and $\lambda_2 = 1$ and every block contains one treatment from each group.

Example 3.3 (Example 3.1 continued) This design is group divisible with groups $1,2 \parallel 3,4 \parallel 5,6$ and concurrences $\lambda_1 = 2$, $\lambda_2 = 1$. ■

Lemma 3.2 *In a partially balanced incomplete-block design,*

$$\sum_{i=1}^{s} a_i\lambda_i = r(k-1).$$

In particular, in a balanced incomplete-block design,

$$(t-1)\lambda = r(k-1).$$

Proof Use Lemma 3.1 (ii). ■

Example 3.4 The design with $t = 4$, $b = 6$, $k = 2$, $r = 3$ and blocks

$$a = \{1,2\}, \quad b = \{1,3\}, \quad c = \{1,4\}, \quad d = \{2,3\}, \quad e = \{2,4\}, \quad f = \{3,4\}$$

is balanced. Its dual has $t = 6$, $b = 4$, $k = 3$, $r = 2$ and blocks

$$1 = \{a,b,c\}, \quad 2 = \{a,d,e\}, \quad 3 = \{b,d,f\}, \quad 4 = \{c,e,f\}.$$

This is group divisible for the groups $a, f \parallel b, e \parallel c, d$. In fact, it is a transversal design. ■

Warning: duals of partially balanced incomplete-block designs are not always partially balanced. I have introduced duals to enlarge the class of designs we can study.

Example 3.5 Let $\Theta = \{0,1,2,3,4,5,6\}$. The blocks $\{1,2,4\}$, $\{2,3,5\}$, $\{3,4,6\}$, $\{4,5,0\}$, $\{5,6,1\}$, $\{6,0,2\}$, $\{0,1,3\}$ form a balanced incomplete-block design with $t = b = 7$, $r = k = 3$ and $\lambda = 1$. ■

A balanced incomplete-block design for which $k = 3$ and $\lambda = 1$ is called a *Steiner triple system*.

Example 3.6 By taking the complement of each block in Example 3.5, we obtain the following balanced incomplete-block design with $t = b = 7$, $r = k = 4$ and $\lambda = 2$. The blocks are $\{0,3,5,6\}$, $\{1,4,6,0\}$, $\{2,5,0,1\}$, $\{3,6,1,2\}$, $\{4,0,2,3\}$, $\{5,1,3,4\}$ and $\{6,2,4,5\}$. ■

Example 3.7 Let Θ consist of all 2-subsets of $\{1,2,3,4,5\}$. Consider the ten blocks of the form

$$\{\{i,j\},\{i,k\},\{j,k\}\}.$$

They form a triangular design. It has $t = b = 10$, $r = k = 3$, $\lambda_1 = 1$ and $\lambda_2 = 0$. ■

Example 3.8 More generally, let Θ consist of all m-subsets of an n-set Γ, where $n \geqslant 2m$. Choose l with $m < l < n$. For each l-subset Φ of Γ, form the block $\{\theta \in \Theta : \theta \subset \Phi\}$. The set of such blocks is an incomplete-block design with $t = {}^{n}\mathrm{C}_{m}$, $b = {}^{n}\mathrm{C}_{l}$, $k = {}^{l}\mathrm{C}_{m}$ and $r = {}^{n-m}\mathrm{C}_{n-l}$. It is partially balanced with respect to the Johnson association scheme $\mathrm{J}(n,m)$.

Example 3.9 Let Θ be the set of 16 cells in a 4×4 square array and let M be any 4×4 Latin square. Construct an incomplete-block design with 12 blocks of size 4 as follows. Each row is a block. Each column is a block. For each letter of each of M, the cells which have that letter form a block.

For example, if

$$\Theta = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline 13 & 14 & 15 & 16 \\ \hline \end{array} \qquad \text{and} \qquad M = \begin{array}{|c|c|c|c|}\hline A & B & C & D \\ \hline B & A & D & C \\ \hline C & D & A & B \\ \hline D & C & B & A \\ \hline \end{array}$$

then the blocks are $\{1,2,3,4\}$, $\{5,6,7,8\}$, $\{9,10,11,12\}$, $\{13,14,15,16\}$, $\{1,5,9,13\}$, $\{2,6,10,14\}$, $\{3,7,11,15\}$, $\{4,8,12,16\}$, $\{1,6,11,16\}$, $\{2,5,12,15\}$, $\{3,8,9,14\}$ and $\{4,7,10,13\}$.

This design is partially balanced with respect to the $\mathrm{L}(3,4)$-type association scheme defined by M. ■