Chapter 3

Partially Balanced Incomplete-Block Designs

3.1 Block designs

Let Ω be a set of experimental units, called 'plots', to which we can apply treatments in an experiment. Suppose that Ω is partitioned into b blocks of k plots each. Define B in $\mathbb{R}^{\Omega\times\Omega}$ by

$$B(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same block} \\ 0 & \text{otherwise.} \end{cases}$$

Then the group divisible association scheme $\mathrm{GD}(b,k)$ on Ω has adjacency matrices I, $B-I$ and $J-B$.

A *block design* on Ω for treatment-set Θ is a function $\phi\colon\Omega\to\Theta$. Strictly speaking, the design is the quadruple $(\Omega,\mathcal{Q},\Theta,\phi)$, where Ω is the set of plots, $\mathcal{Q}$ is the group divisible association scheme on Ω, Θ is the set of treatments and ϕ is the design function from Ω to Θ. We say that "treatment θ occurs on plot ω" if $\phi(\omega)=\theta$. Define the *design matrix* X in $\mathbb{R}^{\Omega\times\Theta}$ by

$$X(\omega,\theta) = \begin{cases} 1 & \text{if } \phi(\omega)=\theta \\ 0 & \text{otherwise.} \end{cases}$$

Let Δ be the set of blocks. The *incidence matrix* is the matrix N in $R^{\Delta\times\Theta}$ given by

$$N(\delta,\theta) = |\{\omega\in\delta : \phi(\omega)=\theta\}|\,.$$

The design is *binary* if $N(\delta,\theta)\in\{0,1\}$ for all δ in Δ and all θ in Θ. In this case we often identify block δ with $\{\theta\in\Theta : N(\delta,\theta)=1\}$ and identify the design with the family of blocks.

The design is a *complete-block design* if it is binary and $k = |\Theta|$, so that $N = J_{\Delta,\Theta}$. It is an *incomplete-block design* if it is binary and $k < |\Theta|$, so that each block is incomplete in the sense of not containing all of the treatments.

(When Yates introduced incomplete-block designs he called them 'incomplete block designs' to contrast them with complete block designs. However, when pure mathematicians took up his term, they assumed that 'incomplete' referred to the *design* and meant that not every k-subset of the treatments was a block. That is why I insist on the hyphen in 'incomplete-block design'.)

The *replication* r_θ of treatment θ is

$$|\{\omega \in \Omega : \phi(\omega) = \theta\}|.$$

The design is *equi-replicate* if all treatments have the same replication. If there are t treatments with replication r then

$$tr = bk. \tag{3.1}$$

In such a design, $X'X = rI$.

The trick for remembering the parameters is that b and k are to do with **b**loc**k**s while t and r concern **tr**eatments. (In his first paper on incomplete-block designs, Yates was considering agricultural field trials to compare different varieties. So he called the treatments *varieties*, and let their number be v. This gives the mnemonic **v**a**r**ieties. In his second paper on the subject he switched to the less specific word *treatment*, and established the notation t, r, b, k used here. Statisticians have tended to follow this second usage, but pure mathematicians have retained v for the number of treatments, even though they often call them *points*.)

The *concurrence* $\Lambda(\theta,\eta)$ of treatments θ and η is

$$|\{(\alpha,\beta) \in \Omega \times \Omega : \alpha,\ \beta \text{ in the same block}, \phi(\alpha) = \theta,\ \phi(\beta) = \eta\}|.$$

The matrix Λ is called the *concurrence matrix*. Equi-replicate block designs in which every non-diagonal concurrence is in $\{0,1\}$ are called *configurations*.

Lemma 3.1 *(i) If the design is binary then $\Lambda(\theta,\theta)$ is equal to the replication of θ, and, for $\eta \neq \theta$, $\Lambda(\theta,\eta)$ is equal to the number of blocks in which θ and η both occur.*

(ii) $\sum_{\eta\in\Theta} \Lambda(\theta,\eta) = kr_\theta$; in particular, in an equi-replicate design

$$\sum_{\substack{\eta\in\Theta \\ \eta\neq\theta}} \Lambda(\theta,\eta) = r(k-1).$$

(iii) $\Lambda = X'BX = N'N$.

Example 3.1 A small design on 12 experimental units is shown below. The set Ω consists of three blocks δ_1, δ_2, δ_3 of four plots each, so $b = 3$ and $k = 4$. The treatment-set Θ is $\{1,2,3,4,5,6\}$, so $t = 6$. The design is equi-replicate with $r = 2$.

	δ_1				δ_2				δ_3			
Ω	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}
ϕ	1	2	3	4	1	2	5	6	3	4	5	6

With the plots in the obvious order, the blocks matrix B is given by

$$B = \begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&1&1&0&0&0&0\\ 0&0&0&0&1&1&1&1&0&0&0&0\\ 0&0&0&0&1&1&1&1&0&0&0&0\\ 0&0&0&0&1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1 \end{bmatrix}.$$

The stratum projectors are $\frac{1}{12}J$, $\frac{1}{4}B - \frac{1}{12}J$ and $I - \frac{1}{4}B$.

With the plots in the same order as above and treatments also in the obvious order, the design matrix X is given by

$$X = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}$$

Then $X'X = 2I$.

The incidence matrix N and concurrence matrix Λ are shown with their rows and columns labelled for clarity.

$$N = \begin{array}{c} \\ \delta_1 \\ \delta_2 \\ \delta_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \end{array} \qquad \Lambda = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 \end{array}\right] \end{array}$$

It is clear that $\Lambda = N'N = X'BX$. ■

The *dual* block design to $(\Omega, \mathcal{Q}, \Theta, \phi)$ is obtained by interchanging the roles of Θ and Δ. So the experimental units of the dual design are still the elements of Ω, but the blocks are the sets $\phi^{-1}(\theta)$ for θ in Θ. These new blocks define a group divisible association scheme $\mathcal{Q}^*$ on Ω of type $\mathrm{GD}(t, r)$. The dual design is the quadruple $(\Omega, \mathcal{Q}^*, \Delta, \psi)$ where the dual design function $\psi\colon \Omega \to \Delta$ is given by

$$\psi(\omega) = \text{old block containing } \omega.$$

The incidence matrix for the dual design is N'.

Example 3.2 The dual of the design in Example 3.1 has treatment-set $\{\delta_1, \delta_2, \delta_3\}$. Its blocks are $\{\delta_1, \delta_2\}$, $\{\delta_1, \delta_2\}$, $\{\delta_1, \delta_3\}$, $\{\delta_1, \delta_3\}$, $\{\delta_2, \delta_3\}$ and $\{\delta_2, \delta_3\}$.

Definition The *treatment-concurrence* graph of an incomplete-block design is the (not necessarily simple) graph whose vertices are the treatments and in which the number of edges from θ to η is $\Lambda(\theta, \eta)$ if $\theta \neq \eta$.

Definition An incomplete-block design is *connected* if its treatment-concurrence graph is connected.