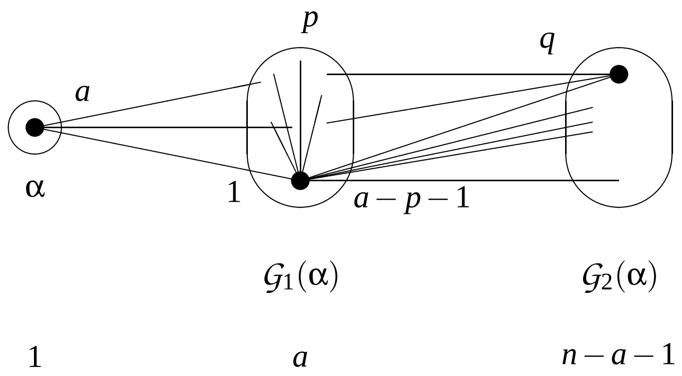
2.5 Parameters of strongly regular graphs

Let $\mathcal{G}$ be a strongly regular graph on n vertices. Put $a_1 = a$, $p^1_{11} = p$ and $p^2_{11} = q$, so that every vertex has valency a, every edge is contained in p triangles, and every non-edge is contained in q paths of length 2.



Counting the edges between $\mathcal{G}_1(\alpha)$ and $\mathcal{G}_2(\alpha)$ in two different ways shows that

$$a(a-p-1) = (n-a-1)q. \tag{2.12}$$

Moreover,

$$A^2 = aI + pA + q(J-A-I),$$

so the eigenvalues of A on the strata other than W_0 are the roots of

$$x^2 + (q-p)x - (a-q) = 0. \tag{2.13}$$

Theorem 2.19 *If the two strata other than W_0 have the same dimension d then $n = 4q+1$, $a = d = 2q$ and $p = q-1$.*

Proof The sum of the roots of Equation (2.13) is equal to $p-q$, and the other eigenvalue is a, so

$$1 + 2d = n$$

and

$$a + d(p-q) = 0.$$

So $d = (n-1)/2$ and d divides a. But $a \leqslant n-2$, because $\mathcal{G}$ is not complete, so $a = d = (n-1)/2$. Then

$$1 + p - q = 0,$$

so $p = q-1$. Substitution in Equation (2.12) gives

$$a(a-q) = (2a+1-a-1)q$$

and so $a = 2q$. ■

Theorem 2.20 *If the two strata other than W_0 have different dimensions then $(q-p)^2+4(a-q)$ is a perfect square t^2, and t divides $(n-1)(p-q)+2a$, and the quotient has the same parity as $n-1$.*

Proof Put $u=(q-p)^2+4(a-q)$. The roots of Equation (2.13) are

$$\frac{p-q\pm\sqrt{u}}{2}.$$

Let d_1 and d_2 be the corresponding dimensions. Then

$$a+d_1\left(\frac{p-q+\sqrt{u}}{2}\right)+d_2\left(\frac{p-q-\sqrt{u}}{2}\right)=0.$$

Since d_1 and d_2 are integers, if they are different then $\sqrt{u}$ is rational. But u is an integer, so it must be a perfect square t^2. Then

$$1+d_1+d_2=n$$

and

$$2a+(d_1+d_2)(p-q)+t(d_1-d_2)=0,$$

whose solutions are

$$\begin{aligned} d_1 &= \frac{1}{2}\left[(n-1)-\frac{(n-1)(p-q)+2a}{t}\right] \\ d_2 &= \frac{1}{2}\left[(n-1)+\frac{(n-1)(p-q)+2a}{t}\right]. \quad \blacksquare \end{aligned}$$

Example 2.7 Let $n=11$ and $a=4$. Then Equation (2.12) gives

$$4(3-p)=6q,$$

whose only solution in non-negative integers is $p=0$, $q=2$. Because 11 is not congruent to 1 modulo 4, Theorem 2.20 applies. But

$$(q-p)^2+4(a-q)=2^2+4(4-2)=12,$$

which is not a perfect square, so there is no strongly regular graph with valency 4 on 11 vertices.

Since interchanging edges and non-edges (taking the complement) of a strongly regular graph yields another strongly regular graph, we see immediately that there is no strongly regular graph with valency 6 on 11 vertices. $\blacksquare$

Note that authors who specialize in strongly regular graphs use n, k, λ and μ where I use n, a, p and q. My choice of notation here is not only for consistency. I can see no need to use two different alphabets for these four parameters. In this subject, k is already over-used, both as a member of the index set $\mathcal{K}$ and as the block size (see Chapter **??**). So is λ, which is not only a general scalar and a general eigenvalue, but which is firmly established as the notation for concurrence for Chapter **??**. And μ is an occasional mate for λ, and more specifically the Möbius function in Chapter **??**.