## 2.4 Techniques

Given an association scheme in terms of its parameters of the first kind, we want to find

- its strata;
- the dimensions of the strata;
- the matrix $D$ expressing the stratum projectors as linear combinations of the adjacency matrices;
- the minimum polynomial of each adjacency matrix;
- the eigenvalues of each adjacency matrix;
- the character table (the matrix $C$).

There are several techniques for doing this.

**Definition** A subspace $W$ of $\mathbb{R}^{\Omega}$ is *invariant* under a matrix $M$ in $\mathbb{R}^{\Omega\times\Omega}$ if $Mw \in W$ for all $w$ in $W$. It is invariant under $\mathcal{A}$ if it is invariant under every matrix in $\mathcal{A}$.

Of course, the strata are invariant under every matrix in the Bose-Mesner algebra, but they are not the only such subspaces, so our first technique needs a little luck.

**Technique 2.1** Use knowledge of or experience of or intuition about the symmetry of the association scheme to find "natural" invariant subspaces. By taking intersections and complements, refine these to a set of $s+1$ mutually orthogonal subspaces, including $W_0$, whose sum is $\mathbb{R}^{\Omega}$. Then verify that each of these subspaces is a sub-eigenspace of each adjacency matrix.

This is the technique we adopted in Example 2.2. If it works it is a marvellous technique, because it gives the most insight into the strata and their dimensions, and it gives the character table (hence all of the eigenvalues) as part of the verification. But this is cold comfort to someone without the insight to guess the strata. The next technique is completely systematic and always works.

**Technique 2.2** Choose one of the adjacency matrices $A_i$ and express its powers in terms of $A_0, \ldots, A_s$, using the equations

$$A_iA_j = \sum_k p_{ij}^k A_k.$$

Hence find the minimum polynomial of $A_i$, which has degree at most $s+1$. Factorize this minimum polynomial to obtain the eigenvalues of $A_i$. If you are lucky, it has $s+1$ distinct eigenvalues $\lambda_0, \ldots, \lambda_s$. Then the eigenspaces of $A_i$ are the strata and the projectors onto them are $S_0, \ldots, S_s$ where

$$S_e = \frac{\prod_{f \neq e}(A_i - \lambda_f I)}{\prod_{f \neq e}(\lambda_e - \lambda_f)}.$$

Expressing these in terms of $A_0, \ldots, A_s$ (we already have the powers of $A_i$ in this form) gives the entries in $D$.

**Example 2.3 (Example 1.5 continued)** In the cube association scheme, write $Y = A_{\text{yellow}}$, $B = A_{\text{black}}$ and $R = A_{\text{red}}$. Then

$$YB = \sum_i p^i_{\text{yellow,black}} A_i = 2Y + 3R.$$

This can be seen by reading off the values of $p^i_{\text{yellow,black}}$ from Example 1.5. Equivalently, start at a point on the cube, take one yellow step and then one black one: where can you get to? The point at the end of the red edge from the starting point can be reached in three ways along a yellow-black path, and each point at the end of a yellow edge from the starting point can be reach in two ways along such a path. Similarly,

$$Y^2 = 3I + 2B$$

and

$$YR = B.$$

So

$$Y^3 = 3Y + 2YB = 3Y + 4Y + 6R = 7Y + 6R$$

and

$$Y^4 = 7Y^2 + 6YR = 7Y^2 + 6B = 7Y^2 + 3(Y^2 - 3I) = 10Y^2 - 9I.$$

Thus

$$O = Y^4 - 10Y^2 + 9I = (Y^2 - 9I)(Y^2 - I),$$

so the minimum polynomial of $Y$ is

$$(X - 3I)(X + 3I)(X - I)(X + I)$$

and the eigenvalues of $Y$ are $\pm 3$ and $\pm 1$. (This verifies one eigenvalue we already know: the valency of yellow, which is 3.)

The eigenspace $W_{+1}$ of $Y$ with eigenvalue $+1$ has projector $S_{+1}$ given by

$$
\begin{aligned}
S_{+1} &= \frac{(Y^2-9I)(Y+I)}{(1^2-9)(1+1)} = \frac{(3I+2B-9I)(Y+I)}{-16} \\
&= \frac{(3I-B)(Y+I)}{8} = \frac{3I+3Y-B-YB}{8} = \frac{3I+Y-B-3R}{8}.
\end{aligned}
$$

The other three eigenprojectors are found similarly. The dimensions follow immediately: for example

$$d_{+1} = \dim W_{+1} = \mathrm{tr}(S_{+1}) = 3. \quad \blacksquare$$

There are two small variations on Technique 2.2. The first is necessary if your chosen $A_i$ has fewer than $s+1$ distinct eigenvalues. Then you need to find the eigenprojectors for at least one more adjacency matrix, and take products of these with those for $A_i$. Continue until you have $s+1$ different non-zero projectors. A good strategy is to choose a complicated adjacency matrix to start with, so that it is likely to have many eigenvalues. That is why we used $Y$ instead of $R$ in Example 2.3.

The other variation is always possible, and saves some work. We know that $\chi_\Omega$ is always an eigenvector of $A_i$ with eigenvalue $a_i$. So we can find the other eigenvalues of $A_i$ by working on the orthogonal complement $W_0^\perp$ of $\chi_\Omega$. We need to find the polynomial $p(A_i)$ of lowest degree such that $p(A_i)v = 0$ for all $v$ in $W_0^\perp$. Every power of $A_i$ has the form $\sum_j \mu_j A_j$. If $v \in W_0^\perp$ then $Jv = 0$ so $(\sum_j A_j)v = 0$, so

$$\sum_{j=0}^{s} \mu_j A_j v = \sum_{j=0}^{s-1} (\mu_j - \mu_s) A_j v.$$

Thus for our calculations we can pretend that $J = O$ and work with one fewer of the $A_j$.

**Example 2.3 revisited** If we put $J = O$ we get $R = -(I+Y+B)$, so

$$Y^3 = 7Y - 6(I+Y+B) = Y - 6I - 3(Y^2 - 3I)$$

so

$$Y^3 + 3Y^2 - Y - 3I = O$$

so

$$(Y^2 - I)(Y + 3I) = O.$$

Now, $a_{\text{yellow}} = 3$ and $Y - 3I$ is not already a factor of this polynomial, so the minimum polynomial is

$$(X - 3I)(X^2 - I)(X + 3I),$$

as before. $\blacksquare$

**Technique 2.3** If $C$ and the dimensions are known, find $D$ by using

$$D = \frac{1}{n}\operatorname{diag}(d)C'\operatorname{diag}(a)^{-1}. \tag{2.5}$$

If $D$ is known, find the dimensions from the first column and then find $C$ by using

$$C = n\operatorname{diag}(a)D'\operatorname{diag}(d)^{-1}. \tag{2.6}$$

**Technique 2.4** Use the orthogonality relations and/or the fact that the dimensions must be integers to complete $C$ or $D$ from partial information. In particular

$$\sum_{i\in\mathcal{K}} C(i,e) = 0 \qquad \text{if } e \neq 0; \tag{2.7}$$

$$\sum_{e\in\mathcal{E}} C(i,e)d_e = 0 \qquad \text{if } i \neq 0; \tag{2.8}$$

$$\sum_{e\in\mathcal{E}} D(e,i) = 0 \qquad \text{if } i \neq 0 \tag{2.9}$$

and

$$\sum_{i\in\mathcal{K}} D(e,i)a_i = 0 \qquad \text{if } e \neq 0. \tag{2.10}$$

**Example 2.4** In the association scheme 5 defined by the 5-circuit, let $A_1$ be the adjacency matrix for edges. Then

$$A_1^2 = 2I + (J - A_1 - I).$$

Ignoring $J$, we have $A_1^2 + A_1 - I = O$, so we find that the eigenvalues of $A_1$ on $W_0^\perp$ are

$$\frac{-1\pm\sqrt{5}}{2}.$$

Let the other two strata be $W_1$ and $W_2$, with dimensions $d_1$ and $d_2$. Then the incomplete character table is

$$\begin{array}{cc c}
 & & \begin{array}{ccc} W_0 & W_1 & W_2 \\ (1) & (d_1) & (d_2) \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{array}{c} (1) \\ (2) \\ (2) \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 2 & \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \\ 2 & & \end{bmatrix}
\end{array}$$

(it is helpful to show the valencies and dimensions in parentheses like this). Then

$$\sum_e C(1,e)d_e = 2 + \left(\frac{-1+\sqrt{5}}{2}\right) d_1 + \left(\frac{-1-\sqrt{5}}{2}\right) d_2 = 0.$$

To get rid of the $\sqrt{5}$, we must have $d_1 = d_2$. But $\sum_e d_e = 5$, so $d_1 = d_2 = 2$.

The sums of the entries in the middle column and the final column must both be 0, so the complete character table is as follows.

$$\begin{array}{cc}
 & \begin{array}{ccc} W_0 & W_1 & W_2 \\ (1) & (2) & (2) \end{array} \\
\begin{array}{cc} 0 & (1) \\ 1 & (2) \\ 2 & (2) \end{array} &
\begin{bmatrix} 1 & 1 & 1 \\ 2 & \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \\ 2 & \frac{-1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \end{bmatrix}
\end{array}$$

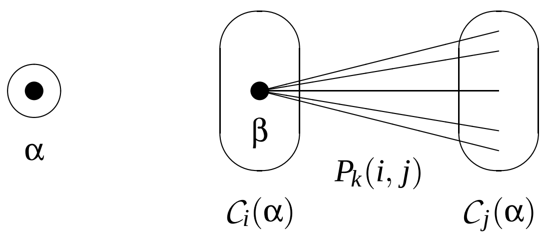
■

There is alternative to Technique 2.2, which uses a remarkable algebra isomorphism.

**Theorem 2.16** *For $k$ in $\mathcal{K}$ define the matrix $P_k$ in $\mathbb{R}^{\mathcal{K}\times\mathcal{K}}$ by*

$$P_k(i,j) = p^i_{jk} = p^i_{kj}$$

*(so that $P_k(i,j)$ is the number of $k$-coloured edges from $\beta$ to $\mathcal{C}_j(\alpha)$ if $\beta \in \mathcal{C}_i(\alpha)$).*




*Let $\mathcal{P} = \left\{ \sum_{i\in\mathcal{K}} \lambda_i P_i : \lambda_0, \ldots, \lambda_s \in \mathbb{R} \right\}$. Define $\varphi\colon \mathcal{A} \to \mathcal{P}$ by $\varphi(A_i) = P_i$, extended linearly. Then $\varphi$ is an algebra isomorphism, called the* Bose-Mesner isomorphism.

**Proof** The set of matrices $\{P_i : i \in \mathcal{K}\}$ is linearly independent, because $P_k(0, j) = 0$ unless $j = k$. Hence it is sufficient to prove that $\varphi(A_iA_j) = \varphi(A_i)\varphi(A_j)$.

Matrix multiplication is associative, so

$$(A_iA_j)A_z = A_i(A_jA_z)$$

for $i$, $j$ and $z$ in $\mathcal{K}$. But

$$(A_iA_j)A_z = \sum_{k\in\mathcal{K}} p_{ij}^k A_kA_z = \sum_{k\in\mathcal{K}}\sum_{x\in\mathcal{K}} p_{ij}^k p_{kz}^x A_x$$

and

$$A_i(A_jA_z) = A_i\left(\sum_{y\in\mathcal{K}} p_{jz}^y A_y\right) = \sum_{x\in\mathcal{K}}\sum_{y\in\mathcal{K}} p_{iy}^x p_{jz}^y A_x.$$

The adjacency matrices are linearly independent, so

$$\sum_k p_{ij}^k p_{kz}^x = \sum_y p_{iy}^x p_{jz}^y \tag{2.11}$$

for $i$, $j$, $x$ and $z$ in $\mathcal{K}$.

Now

$$\begin{aligned}(P_iP_j)(x,z) &= \sum_y P_i(x,y)P_j(y,z)\\ &= \sum_y p_{iy}^x p_{jz}^y\\ &= \sum_k p_{ij}^k p_{kz}^x\\ &= \sum_k p_{ij}^k P_k(x,z)\end{aligned}$$

so $P_iP_j = \sum_k p_{ij}^k P_k$ and thus $\varphi(A_i)\varphi(A_j) = \sum_k p_{ij}^k \varphi(A_k) = \varphi(\sum_k p_{ij}^k A_k) = \varphi(A_iA_j)$. ■

**Corollary 2.17** *The matrices $A_i$ and $P_i$ have the same minimum polynomial and hence the same eigenvalues.*

**Technique 2.5** Working in $\mathbb{R}^{\mathcal{K}\times\mathcal{K}}$, find the eigenvalues and minimum polynomial of $P_i$. These are the eigenvalues and minimum polynomial of $A_i$.

**Example 2.5 (Example 1.4 continued)** In the Petersen graph,

$$P_1 = \begin{bmatrix} 0 & 3 & 0\\ 1 & 0 & 2\\ 0 & 1 & 2 \end{bmatrix}.$$

By inspection,

$$P_1 + 2I = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 4 \end{bmatrix},$$

which is singular, so $-2$ is an eigenvalue. We know that 3 is an eigenvalue, because $a_1 = 3$. The sum of the eigenvalues is equal to $\mathrm{tr}(P_1)$, which is 2, so the third eigenvalue is 1.

Let $d_1$ and $d_2$ be the dimensions of the strata corresponding to eigenvalues $-2$, 1 respectively. Then

$$\sum_e d_e = 1 + d_1 + d_2 = 10$$

and

$$\sum_e C(1,e) d_e = 3 - 2d_1 + d_2 = 0,$$

so $d_1 = 4$ and $d_2 = 5$. Now the incomplete character table is

$$\begin{array}{cc|c} & & \begin{array}{ccc} 0 & 1 & 2 \\ (1) & (4) & (5) \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{array}{c} (1) \\ (3) \\ (6) \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 3 & -2 & 1 \\ 6 & & \end{bmatrix} \end{array}$$

Apart from the 0-th column, the column sums must be zero, so

$$C = \begin{array}{cc|c} & & \begin{array}{ccc} 0 & 1 & 2 \\ (1) & (4) & (5) \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{array}{c} (1) \\ (3) \\ (6) \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 3 & -2 & 1 \\ 6 & 1 & -2 \end{bmatrix} \end{array}.$$

Then

$$\begin{aligned} D &= \frac{1}{10} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 6 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 1 & 1 & 1 \\ 4 & -\frac{8}{3} & \frac{2}{3} \\ 5 & \frac{5}{3} & -\frac{5}{3} \end{bmatrix}. \quad \blacksquare \end{aligned}$$

**Theorem 2.18** *In the cyclic association scheme defined by the blueprint* $\{\Delta_i : i \in \mathcal{K}\}$ *of* $\mathbb{Z}_n$*, the eigenvalues of* $A_i$ *are*

$$\sum_{\alpha\in\Delta_i} \eta^\alpha$$

*as* $\eta$ *ranges over the complex nth roots of unity.*

**Proof** Consider the complex vector $v = \sum_{\beta\in\Omega} \eta^\beta \chi_\beta$, where $\eta^n = 1$. Here $A_i = \sum_{\alpha\in\Delta_i} M_\alpha$, so

$$\begin{aligned} A_i v &= \sum_{\alpha\in\Delta_i} M_\alpha \sum_{\beta\in\Omega} \eta^\beta \chi_\beta \\ &= \sum_{\alpha\in\Delta_i} \sum_{\beta\in\Omega} \eta^\beta M_\alpha \chi_\beta \\ &= \sum_{\alpha\in\Delta_i} \eta^\alpha \sum_{\beta\in\Omega} \eta^{\beta-\alpha} \chi_{\beta-\alpha} \\ &= \sum_{\alpha\in\Delta_i} \eta^\alpha v. \end{aligned}$$

Note that addition is modulo $n$ in both $\eta^{\beta-\alpha}$ and $\chi_{\beta-\alpha}$, so there is no problem about inconsistency. Now, $\Delta_i = -\Delta_i$, so not only is

$$\sum_{\alpha\in\Delta_i} \eta^{-\alpha} = \sum_{\alpha\in\Delta_i} \eta^\alpha$$

but also this value is real. If $\eta \in \{1, -1\}$ then $v$ is real; otherwise $v$ and its complex conjugate $\bar{v}$ have the same real eigenvalue so $v + \bar{v}$ and $i(v - \bar{v})$ are distinct real vectors with eigenvalue $\sum_{\alpha\in\Delta_i} \eta^\alpha$. ■

**Technique 2.6** For a cyclic association scheme on $\mathbb{Z}_n$, calculate the eigenvalues $\sum_{\alpha\in\Delta_i} \eta^\alpha$, where $\eta^n = 1$, and amalgamate those spaces which have the same eigenvalue on every adjacency matrix.

**Example 2.6** The 6-circuit gives the blueprint $\{0\}$, $\{\pm 1\}$, $\{\pm 2\}$, $\{3\}$ of $\mathbb{Z}_6$. Then $A_1 = M_1 + M_5$, so the eigenvalues of $A_1$ are

$$\begin{array}{ll} \theta + \theta^5 & \text{(twice)} \\ \theta^2 + \theta^4 & \text{(twice)} \\ \theta^3 + \theta^3 & \text{(once)} \\ \theta^6 + \theta^6 & \text{(once),} \end{array}$$

where $\theta$ is a primitive sixth root of unity in $\mathbb{C}$. But $\theta^6 = 1$, $\theta^3 = -1$, $\theta^2 + \theta^4 = -1$ (because the cube roots of unity sum to zero) and $\theta + \theta^5 = 1$ (because the sixth roots of unity sum to zero). So a portion of the character table is

$$\begin{array}{cc}
 & \begin{array}{cccc} \theta^6 & \theta^{\pm 1} & \theta^{\pm 2} & \theta^3 \\ (1) & (2) & (2) & (1) \end{array} \\
\begin{array}{rc} 0 & (1) \\ \pm 1 & (2) \\ \pm 2 & (2) \\ 3 & (1) \end{array} & \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ 2 & & & \\ 2 & & & \end{array}\right]
\end{array}.$$

In this case the matrix is already square, so there is no amalgamation of columns. If there are fewer than $\lceil (n+1)/2 \rceil$ columns, complete the table and then amalgamate identical columns. ■

The techniques are summarized in Figure 2.1.

Note that many authors use $P$ and $Q$ for the matrices that I call $C$ and $nD$. Delsarte established this notation in his important work on the connection between association schemes and error-correcting codes, and it was popularized by MacWilliams and Sloane's book. However, I think that $P$ and $Q$ are already overused in this subject. Quite apart from the use of $P$ for probability in connection with random responses in designed experiments (see Chapter **??**), $P$ and $Q$ are well established notation for projectors, and $P$ is also the obvious letter for the matrices in Theorem 2.16. Moreover, I find $C$ and $D$ more memorable, because $C$ contains the **c**haracters while $D$ contains the **d**imensions.

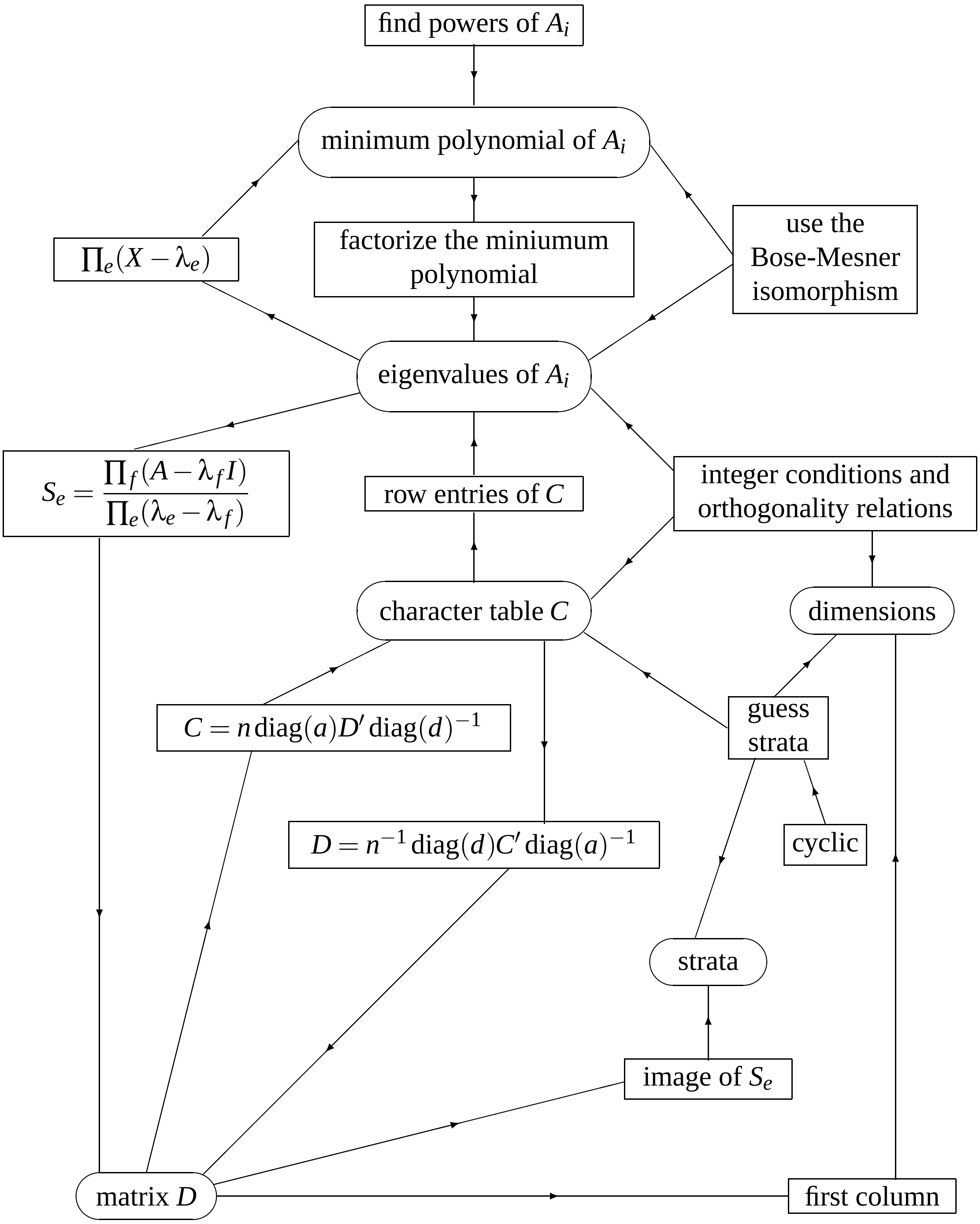



Figure 2.1: Techniques for finding parameters of the second kind