Theorem 2.6 *Let $\mathcal{A}$ be the Bose-Mesner algebra of an association scheme on Ω with s associate classes and adjacency matrices $A_0, A_1, \ldots, A_s$. Then $\mathbb{R}^\Omega$ has $s+1$ mutually orthogonal subspaces W_0, W_1, ..., W_s, called* strata, *with orthogonal projectors $S_0, S_1, \ldots, S_s$ such that*

(i) $\mathbb{R}^\Omega = W_0 \oplus W_1 \oplus \cdots \oplus W_s$;

(ii) each of $W_0, W_1, \ldots, W_s$ is a sub-eigenspace of every matrix in $\mathcal{A}$;

(iii) for $i = 0, 1, \ldots, s$, the adjacency matrix A_i is a linear combination of S_0, S_1, ..., S_s;

(iv) for $e = 0, 1, \ldots, s$, the stratum projector S_e is a linear combination of A_0, $A_1, \ldots, A_s$.

Proof The adjacency matrices A_1, ..., A_s commute and are symmetric, so $s-1$ applications of Lemma 2.4, starting with the eigenspaces of A_1, give spaces W_0, ..., W_r as the non-zero intersections of the eigenspaces of A_1, ..., A_s, where r is as yet unknown. These spaces W_e are mutually orthogonal and satisfy (i). Since $A_0 = I$ and every matrix in $\mathcal{A}$ is a linear combination of $A_0, A_1, \ldots, A_s$, the spaces W_e clearly satisfy (ii). Each S_e is a polynomial in $A_1, \ldots, A_s$, hence in $\mathcal{A}$, so (iv) is satisfied. Let $C(i,e)$ be the eigenvalue of A_i on W_e. Then Equation (2.2) shows that

$$A_i = \sum_{e=0}^{r} C(i,e) S_e$$

and so (iii) is satisfied.

Finally, (iii) shows that S_0, ..., S_r span $\mathcal{A}$. Suppose that there are real scalars $\lambda_0, \ldots, \lambda_r$ such that $\sum_e \lambda_e S_e = O$. Then for $f = 0, \ldots, r$ we have $O = (\sum_e \lambda_e S_e) S_f = \lambda_f S_f$ so $\lambda_f = 0$. Hence S_0, ..., S_r are linearly independent, so they form a basis for $\mathcal{A}$. Thus $r+1 = \dim \mathcal{A} = s+1$ and $r = s$. ■

We now have two bases for $\mathcal{A}$: the adjacency matrices and the stratum projectors. The former are useful for addition, because $A_j(\alpha,\beta) = 0$ if $(\alpha,\beta) \in \mathcal{C}_i$ and $i \neq j$. The stratum projectors make multiplication easy, because $S_e S_e = S_e$ and $S_e S_f = O$ if $e \neq f$.

Before calculating any eigenvalues, we note that if A is the adjacency matrix of any subset $\mathcal{C}$ of $\Omega \times \Omega$ then $A\chi_\alpha = \sum \{\chi_\beta : (\beta,\alpha) \in \mathcal{C}\}$. In particular, $A_i \chi_\alpha = \chi_{\mathcal{C}_i(\alpha)}$ and $J\chi_\alpha = \chi_\Omega$. Furthermore, if M is any matrix in $\mathbb{R}^{\Omega\times\Omega}$ then $M\chi_\Omega = \sum_{\omega\in\Omega} M\chi_\omega$. If M has constant row-sum r then $M\chi_\Omega = r\chi_\Omega$.

Example 2.2 Consider the group divisible association scheme $\mathrm{GD}(b,k)$ with b groups of size k and

$$A_1(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same group but } \alpha \neq \beta \\ 0 & \text{otherwise} \end{cases}$$
$$A_2(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in different groups} \\ 0 & \text{otherwise.} \end{cases}$$

Consider the 1-dimensional space W_0 spanned by χ_Ω. We have

$$\begin{aligned} A_0\chi_\Omega = I_\Omega\chi_\Omega &= \chi_\Omega \\ A_1\chi_\Omega = a_1\chi_\Omega &= (k-1)\chi_\Omega \\ A_2\chi_\Omega = a_2\chi_\Omega &= (b-1)k\chi_\Omega. \end{aligned}$$

Thus W_0 is a sub-eigenspace of every adjacency matrix. This does not prove that W_0 is a stratum, because there might be other vectors which are also eigenvectors of the all the adjacency matrices with the same eigenvalues as W_0.

Let α and β be in the same group Δ. Then

$$\begin{aligned} A_1\chi_\alpha &= \chi_\Delta - \chi_\alpha & \qquad A_2\chi_\alpha &= \chi_\Omega - \chi_\Delta \\ A_1\chi_\beta &= \chi_\Delta - \chi_\beta & \qquad A_2\chi_\beta &= \chi_\Omega - \chi_\Delta \end{aligned}$$

so $A_1(\chi_\alpha - \chi_\beta) = -(\chi_\alpha - \chi_\beta)$ and $A_2(\chi_\alpha - \chi_\beta) = 0$. Let W_{within} be the $b(k-1)$-dimensional "within groups" subspace spanned by all vectors of the form $\chi_\alpha - \chi_\beta$ with α and β in the same group. Then W_{within} is a sub-eigenspace of A_0, A_1 and A_2 and the eigenvalues for A_1 and A_2 are different from those for W_0.

Since eigenspaces are mutually orthogonal, it is natural to look at the orthogonal complement of $W_0 + W_{\text{within}}$. This is the $(b-1)$-dimensional "between groups" subspace W_{between} spanned by vectors of the form $\chi_\Delta - \chi_\Gamma$ where Δ and Γ are different groups. Now

$$A_1\chi_\Delta = A_1 \sum_{\alpha\in\Delta} \chi_\alpha = k\chi_\Delta - \sum_{\alpha\in\Delta} \chi_\alpha = (k-1)\chi_\Delta$$

and

$$A_2\chi_\Delta = A_2 \sum_{\alpha\in\Delta} \chi_\alpha = k(\chi_\Omega - \chi_\Delta)$$

so $A_1(\chi_\Delta - \chi_\Gamma) = (k-1)(\chi_\Delta - \chi_\Gamma)$ and $A_2(\chi_\Delta - \chi_\Gamma) = -k(\chi_\Delta - \chi_\Gamma)$. Thus W_{between} is a sub-eigenspace with different eigenvalues from W_{within}. Therefore the strata are W_0, W_{within} and W_{between}. ■

Lemma 2.7 *If $P \in \mathcal{A}$ and P is idempotent then $P = \sum_{e\in\mathcal{F}} S_e$ for some subset $\mathcal{F}$ of* $\{0,\dots,s\}$.

Proof Let $P = \sum_{e=0}^{s} \lambda_e S_e$. Then $P^2 = \sum_{e=0}^{s} \lambda_e^2 S_e$, which is equal to P if and only if $\lambda_e \in \{0,1\}$ for $e = 0, \dots, s$. ■

For this reason the stratum projectors are sometimes called *minimal idempotents* or *primitive idempotents*.

Lemma 2.8 *The space W spanned by χ_Ω is always a stratum. Its projector is* $|\Omega|^{-1} J_\Omega$.

Proof The orthogonal projector onto W is $|\Omega|^{-1} J_\Omega$ because

$$J_\Omega \chi_\Omega = \sum_{\omega\in\Omega} J_\Omega \chi_\omega = |\Omega|\, \chi_\Omega$$

and

$$J_\Omega(\chi_\alpha - \chi_\beta) = 0.$$

This is an idempotent contained in $\mathcal{A}$, so it is equal to $\sum_{e\in\mathcal{F}} S_e$, for some subset $\mathcal{F}$ of $\{0,\dots,s\}$, by Lemma 2.7. Then

$$1 = \dim W = \operatorname{tr}\left(|\Omega|^{-1} J_\Omega\right) = \sum_{e\in\mathcal{F}} \operatorname{tr} S_e = \sum_{e\in\mathcal{F}} \dim W_e$$

so we must have $|\mathcal{F}| = 1$ and W is itself a stratum. ■

Notation The 1-dimensional stratum spanned by χ_Ω is always called W_0.

Although there are the same number of strata as associate classes, there is usually no natural bijection between them. When I want to emphasize this, I shall use a set $\mathcal{K}$ to index the associate classes and a set $\mathcal{E}$ to index the strata. However, there are some association schemes for which $\mathcal{E}$ and $\mathcal{K}$ are naturally the same but for which W_0 does not correspond to A_0. So the reader should interpret these two subscripts ‘0’ as different sorts of zero.

I shall always write d_e for $\dim W_e$.

2.3 The character table

For i in $\mathcal{K}$ and e in $\mathcal{E}$ let $C(i,e)$ be the eigenvalue of A_i on W_e and let $D(e,i)$ be the coefficient of A_i in the expansion of S_e as a linear combination of the adjacency matrices. That is:

$$A_i = \sum_{e\in\mathcal{E}} C(i,e) S_e \tag{2.3}$$

and

$$S_e = \sum_{i\in\mathcal{K}} D(e,i)A_i. \tag{2.4}$$

Lemma 2.9 *The matrices C in $\mathbb{R}^{\mathcal{K}\times\mathcal{E}}$ and D in $\mathbb{R}^{\mathcal{E}\times\mathcal{K}}$ are mutual inverses.*

We note some special values of $C(i,e)$ and $D(e,i)$:

$$\begin{array}{ll}
C(0,e) = 1 & \text{because } A_0 = I = \sum_{e\in\mathcal{E}} S_e; \\
C(i,0) = a_i & \text{because } A_i\chi_\Omega = a_i\chi_\Omega; \\
D(0,i) = \dfrac{1}{|\Omega|} & \text{because } S_0 = \dfrac{1}{|\Omega|}J = \dfrac{1}{|\Omega|}\sum_{i\in\mathcal{K}} A_i; \\
D(e,0) = \dfrac{d_e}{|\Omega|} & \text{because } d_e \;=\; \mathrm{tr}(S_e) \;=\; \sum_{i\in\mathcal{K}} D(e,i)\,\mathrm{tr}(A_i) \\
 & \qquad\qquad = |\Omega|\,D(e,0).
\end{array}$$

Lemma 2.10 *The map $\varphi_e\colon\mathcal{A}\to\mathcal{A}$ defined by*

$$\varphi_e\colon A_i \mapsto C(i,e)S_e$$

and extended linearly is an algebra homomorphism.

Corollary 2.11 *The maps $\vartheta_0,\ldots,\vartheta_s\colon\mathcal{A}\to\mathbb{R}$ defined by*

$$\vartheta_e\colon \sum_{i\in\mathcal{K}} \lambda_i A_i \mapsto \sum_{i\in\mathcal{K}} \lambda_i C(i,e)$$

are algebra homomorphisms.

Definition The maps ϑ_0, …, ϑ_s are *characters* of the association scheme. The matrix C, whose columns are the characters, is the *character table* of the association scheme.

Example 2.2 revisited The character table is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{c} \begin{array}{ccc} 0 & \text{within} & \text{between} \end{array} \\
\left[\begin{array}{ccc} 1 & 1 & 1 \\ k-1 & -1 & k-1 \\ (b-1)k & 0 & -k \end{array}\right] \end{array}$$

The entries in the 0-th row are are equal to 1; those in the 0-th column are the valencies. ■

Theorem 2.12 (Orthogonality relations for the associate classes)

$$\sum_{e\in\mathcal{E}} C(i,e)C(j,e)d_e = \begin{cases} a_i\,|\Omega| & \textit{if } i=j \\ 0 & \textit{otherwise.} \end{cases}$$

Proof We calculate the trace of A_iA_j in two different ways. First

$$\begin{aligned} A_iA_j &= \left(\sum_e C(i,e)S_e\right)\left(\sum_f C(j,f)S_f\right) \\ &= \sum_e C(i,e)C(j,e)S_e \end{aligned}$$

so

$$\begin{aligned} \operatorname{tr}(A_iA_j) &= \sum_e C(i,e)C(j,e)\operatorname{tr}(S_e) \\ &= \sum_e C(i,e)C(j,e)d_e. \end{aligned}$$

But $A_iA_j = \sum_k p_{ij}^k A_k$ so $\operatorname{tr}(A_iA_j) = p_{ij}^0\,|\Omega|$; and $p_{ij}^0 = 0$ if $i \neq j$, while $p_{ii}^0 = a_i$. ■

Corollary 2.13 *If* $|\Omega| = n$ *then*

$$D = \frac{1}{n}\operatorname{diag}(d)C'\operatorname{diag}(a)^{-1}.$$

Proof The equation in Theorem 2.12 can be written as

$$C\operatorname{diag}(d)C' = n\operatorname{diag}(a)$$

so

$$C\operatorname{diag}(d)C'\operatorname{diag}(a)^{-1} = nI.$$

But $D = C^{-1}$ so $D = n^{-1}\operatorname{diag}(d)C'\operatorname{diag}(a)^{-1}$. ■

Thus C is inverted by transposing it, multiplying the rows by the dimensions, dividing the columns by the valencies, and finally dividing all the entries by the size of Ω.

Example 2.2 revisited Here $n = bk$,

$$\operatorname{diag}(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & (b-1)k \end{bmatrix}$$

and

$$\operatorname{diag}(d) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b(k-1) & 0 \\ 0 & 0 & b-1 \end{bmatrix}$$

so

$$\begin{aligned} D &= \frac{1}{bk}\begin{bmatrix} 1 & 0 & 0 \\ 0 & b(k-1) & 0 \\ 0 & 0 & b-1 \end{bmatrix}\begin{bmatrix} 1 & k-1 & (b-1)k \\ 1 & -1 & 0 \\ 1 & k-1 & -k \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{k-1} & 0 \\ 0 & 0 & \frac{1}{(b-1)k} \end{bmatrix} \\ &= \frac{1}{bk}\begin{bmatrix} 1 & 1 & 1 \\ b(k-1) & -b & 0 \\ b-1 & b-1 & -1 \end{bmatrix}. \end{aligned}$$

Note that the entries in the top row are all equal to $1/bk$, while those in the first column are the dimensions divided by bk.

From D we can read off the stratum projectors as

$$S_0 = \frac{1}{bk}(A_0 + A_1 + A_2) = \frac{1}{bk}J,$$

$$S_{\text{within}} = \frac{1}{bk}\left(b(k-1)A_0 - bA_1\right) = I - \frac{1}{k}(A_0 + A_1) = I - \frac{1}{k}G,$$

where $G = A_0 + A_1$, which is the adjacency matrix for the relation “is in the same group as”, and

$$\begin{aligned} S_{\text{between}} &= \frac{1}{bk}\left((b-1)(A_0 + A_1) - A_2\right) \\ &= \frac{1}{bk}\left((b-1)G - (J-G)\right) = \frac{1}{k}G - \frac{1}{bk}J. \quad \blacksquare \end{aligned}$$

Corollary 2.14 (Orthogonality relations for the characters)

$$\sum_{i \in \mathcal{K}} \frac{C(i,e)C(i,f)}{a_i} = \begin{cases} \dfrac{|\Omega|}{d_e} & \text{if } e = f \\ 0 & \text{otherwise.} \end{cases}$$

Proof Let $n = |\Omega|$. Now $DC = I$ so

$$\frac{1}{n}\operatorname{diag}(d)C' \operatorname{diag}(a)^{-1}C = I$$

so

$$C' \operatorname{diag}(a)^{-1}C = n \operatorname{diag}(d)^{-1},$$

as required. $\blacksquare$

Corollary 2.15 (Orthogonality relations for D)

(i) $$\sum_{i\in\mathcal{K}} D(e,i)D(f,i)a_i = \begin{cases} \dfrac{d_e}{n} & \textit{if } e=f \\ 0 & \textit{otherwise;} \end{cases}$$

(ii) $$\sum_{e\in\mathcal{E}} \frac{D(e,i)D(e,j)}{d_e} = \begin{cases} \dfrac{1}{na_i} & \textit{if } i=j \\ 0 & \textit{otherwise.} \end{cases}$$

The entries in the matrices C and D are called *parameters of the second kind*.