# Chapter 2

# The Bose-Mesner Algebra

## 2.1 Orthogonality

This section gives a brief coverage of those aspects of orthogonality that are necessary to appreciate the Bose-Mesner algebra. I hope that most readers have seen most of this material before, even if in a rather different form.

There is a natural inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^\Omega$ defined by

$$\langle f,g\rangle = \sum_{\omega\in\Omega} f(\omega)g(\omega) \qquad \text{for } f \text{ and } g \text{ in } \mathbb{R}^\Omega.$$

Vectors $f$ and $g$ are *orthogonal* to each other (written $f \perp g$) if $\langle f,g\rangle = 0$. If $U$ and $W$ are subspaces of $\mathbb{R}^\Omega$ with $\langle u,w\rangle = 0$ for all $u$ in $U$ and all $w$ in $W$, we say that $U$ is orthogonal to $W$ (written $U \perp W$.)

If $W$ is a subspace of $\mathbb{R}^\Omega$, the *orthogonal complement* of $W$ is defined by

$$W^\perp = \left\{ v \in \mathbb{R}^\Omega : \langle v,w\rangle = 0 \text{ for all } w \text{ in } W \right\}.$$

Here are some standard facts about orthogonal complements.

- $W^\perp$ is a subspace of $\mathbb{R}^\Omega$;
- $\dim W^\perp + \dim W = \dim \mathbb{R}^\Omega = |\Omega|$;
- $\left(W^\perp\right)^\perp = W$ (recall that we are assuming that $|\Omega|$ is finite);
- $(U+W)^\perp = U^\perp \cap W^\perp$ (recall that $U+W = \{u+w : u \in U \text{ and } w \in W\}$, which is called the *vector space sum* of $U$ and $W$);
- $(U \cap W)^\perp = U^\perp + W^\perp$;

- $\mathbb{R}^\Omega = W \oplus W^\perp$, which is called the *direct sum* of $W$ and $W^\perp$: this means that if $v \in \mathbb{R}^\Omega$ then there are unique vectors $w \in W$ and $u \in W^\perp$ with $v = w + u$.

The last fact enables us to make the following definition.

**Definition** The map $P\colon \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ defined by

$$Pv \in W \quad \text{and} \quad v - Pv \in W^\perp$$

is called the *orthogonal projector* onto $W$.

To show that $P$ is the orthogonal projector onto $W$ it is enough to show that $Pv = v$ for $v$ in $W$ and $Pv = 0$ for $v$ in $W^\perp$.

The orthogonal projector $P$ is a linear transformation. We shall identify it with the matrix that represents it with respect to the basis $\{\chi_\omega : \omega \in \Omega\}$. The phrase 'orthogonal projector' will frequently be abbreviated to 'projector'.

**Lemma 2.1** *Let $W$ and $U$ be subspaces of $\mathbb{R}^\Omega$ with orthogonal projectors $P$ and $Q$ respectively. Then*

*(i)* $P^2 = P$ *(that is, $P$ is* idempotent*);*

*(ii)* $\langle P\chi_\alpha, \chi_\beta \rangle = \langle \chi_\alpha, P\chi_\beta \rangle$ *and hence $P$ is symmetric;*

*(iii)* $\dim W = \operatorname{tr}(P)$*;*

*(iv) if $U = W^\perp$ then $Q = I_\Omega - P$;*

*(v) if $U \perp W$ then $PQ = QP = O_\Omega$.*

Commuting projectors have an importance beyond spaces that are actually orthogonal to each other.

**Definition** Let $W$ and $U$ be subspaces of $\mathbb{R}^\Omega$ with orthogonal projectors $P$ and $Q$ respectively. Then $W$ and $U$ are *geometrically orthogonal* to each other if $PQ = QP$.

**Lemma 2.2** *If $W$ and $U$ are geometrically orthogonal then*

*(i) $PQ$ is the orthogonal projector onto $W \cap U$;*

*(ii)* $\left(W \cap (W \cap U)^\perp\right) \perp U$.

**Proof** (i) We need to prove that $PQ$ is the identity on $W \cap U$ and maps everything in $(W \cap U)^\perp$ to the zero vector. If $v \in W \cap U$ then $PQv = Pv = v$. If $v \in (W \cap U)^\perp = W^\perp + U^\perp$ then $v = x + y$ with $x$ in $W^\perp$ and $y$ in $U^\perp$ so $PQv = PQx + PQy = QPx + PQy = 0$ because $Px = Qy = 0$. (This '0' is the zero vector in $\mathbb{R}^\Omega$.)

(ii) If $v \in W \cap (W \cap U)^\perp$ then $Pv = v$ and $PQv = 0$ so $Qv = QPv = PQv = 0$ so $v \in U^\perp$. ■

Property (ii) gives some clue to the strange name 'geometric orthogonality'. The space $W \cap (W \cap U)^\perp$ is the orthogonal complement of $W \cap U$ inside $W$, and $U \cap (W \cap U)^\perp$ is the orthogonal complement of $W \cap U$ inside $U$. So geometric orthogonality means that these two complements are orthogonal to each other as well as to $W \cap U$:

$$\left(W \cap (W \cap U)^\perp\right) \perp \left(U \cap (W \cap U)^\perp\right).$$

Thus subspaces $W$ and $U$ are geometrically orthogonal if they are as orthogonal as they can be given that they have a non-zero intersection.

**Example 2.1** In $\mathbb{R}^3$, let $W$ be the $xy$-plane and let $U$ be the $xz$-plane. Then $W \cap U$ is the $x$-axis; $W \cap (W \cap U)^\perp$ is the $y$-axis; and $U \cap (W \cap U)^\perp$ is the $z$-axis. So the $xy$-plane is geometrically orthogonal to the $xz$-plane.

On the other hand, the $x$-axis is not geometrically orthogonal to the plane $x = y$. ■

**Lemma 2.3** *Let $W_1, \ldots, W_r$ be subspaces of $\mathbb{R}^\Omega$ with orthogonal projectors $P_1, \ldots, P_r$ which satisfy*

*(i)* $\displaystyle\sum_{i=1}^{r} P_i = I_\Omega$;

*(ii)* $P_iP_j = O_\Omega$ *if* $i \neq j$.

*Then $\mathbb{R}^\Omega$ is the direct sum $W_1 \oplus W_2 \oplus \cdots \oplus W_r$.*

**Proof** We need to show that every vector $v$ in $\mathbb{R}^\Omega$ has a unique expression as a sum of vectors in the $W_i$. Let $v_i = P_i v$. Then

$$v = Iv = \left(\sum P_i\right) v = \sum (P_i v) = \sum v_i$$

with $v_i$ in $W_i$, so $\mathbb{R}^\Omega = W_1 + W_2 + \cdots + W_r$.

For uniqueness, suppose that $v = \sum w_i$ with $w_i$ in $W_i$. Then

$$v_j = P_j v = P_j \left(\sum_i w_i\right) = P_j \left(\sum_i P_i w_i\right) = \left(\sum_i P_j P_i w_i\right) = P_j P_j w_j = w_j. \quad ■$$

## 2.2 The algebra

Given an association scheme on $\Omega$ with $s$ associate classes and adjacency matrices $A_0, A_1, \ldots, A_s$, let

$$\mathcal{A} = \left\{ \sum_{i=0}^{s} \lambda_i A_i : \lambda_0,\ \ldots,\ \lambda_s \in \mathbb{R} \right\}.$$

It is clear that the adjacency matrices are linearly independent, for if $(\alpha,\beta) \in \mathcal{C}_j$ then

$$\left( \sum_i \lambda_i A_i \right) (\alpha,\beta) = \lambda_j.$$

Therefore $\mathcal{A}$ has dimension $s+1$ as a vector space over $\mathbb{R}$. It is closed under multiplication, because of Equation (1.1), so it is an algebra. It is called the *Bose-Mesner algebra*.

If $M \in \mathcal{A}$ then $M$ is symmetric, because every adjacency matrix is symmetric. By a standard result of linear algebra, $M$ is diagonalizable over $\mathbb{R}$. This means that it has distinct real eigenvalues $\lambda_1, \ldots, \lambda_r$ (say, because we do not know the value of $r$) with eigenspaces $W_1, \ldots, W_r$ (this means that $W_j = \left\{ v \in \mathbb{R}^\Omega : Mv = \lambda_j v \right\}$) such that

- $R^\Omega = W_1 \oplus W_2 \oplus \cdots \oplus W_r$;
- the minimum polynomial of $M$ is $\prod_{j=1}^{r} (X - \lambda_j)$, with no repeated factors.

In fact, not only is $M$ diagonalizable but also $W_i \perp W_j$ if $i \neq j$.

The orthogonal projector $P_i$ onto $W_i$ is given by

$$P_i = \frac{\displaystyle\prod_{j \neq i} (M - \lambda_j I)}{\displaystyle\prod_{j \neq i} (\lambda_i - \lambda_j)}, \tag{2.1}$$

for if $v \in W_i$ then

$$\left( \prod_{j \neq i} (M - \lambda_j I) \right) v = \left( \prod_{j \neq i} (\lambda_i - \lambda_j) \right) v$$

but if $v \in W_k$ with $k \neq i$ then

$$\left( \prod_{j \neq i} (M - \lambda_j I) \right) v = 0.$$

Because $\mathcal{A}$ is an algebra containing $I$, $\mathcal{A}$ contains all non-negative powers of $M$, so $\mathcal{A}$ contains the projectors $P_1, \ldots, P_r$.

Note also that

$$M = \sum_{i=1}^{r} \lambda_i P_i. \tag{2.2}$$

If none of the eigenvalues is zero then $M$ is invertible and

$$M^{-1} = \sum_{i=1}^{r} \frac{1}{\lambda_i} P_i,$$

which is also in $\mathcal{A}$.

We need to consider something more general than an inverse. Suppose that $f$ is any function from any set $S$ to any set $T$. Then a function $g\colon T \to S$ is called a *generalized inverse* of $f$ if $fgf = f$ and $gfg = g$. It is evident that set functions always have generalized inverses: if $y \in \operatorname{Im}(f)$ then let $g(y)$ be any element $x$ of $S$ with $f(x) = y$; if $y \notin \operatorname{Im}(f)$ then $g(y)$ may be any element of $S$. If $f$ is invertible then $g = f^{-1}$; otherwise there are several possibilities for $g$. However, when there is more structure then generalized inverses may not exist. Group homomorphisms do not always have generalized inverses; nor do continuous functions in topology. However, linear transformations *do* have generalized inverses.

If $M$ is diagonalizable then one of its generalized inverses has a special form. It is called the *Drazin* inverse or *Moore-Penrose* generalized inverse and written $M^-$. This given by

$$M^- = \sum_{\substack{i=1 \\ \lambda_i \neq 0}}^{r} \frac{1}{\lambda_i} P_i.$$

If $M \in \mathcal{A}$ then $M^- \in \mathcal{A}$.

That concludes the properties of a single element of $\mathcal{A}$. The key fact for dealing with two or more matrices in $\mathcal{A}$ is that $\mathcal{A}$ is *commutative*; that is, if $M \in \mathcal{A}$ and $N \in \mathcal{A}$ then $MN = NM$. This follows from Lemma 1.2. The following rather technical lemma is the building block for the main result.

**Lemma 2.4** *Let $U_1, \ldots, U_m$ be non-zero orthogonal subspaces of $\mathbb{R}^\Omega$ such that $\mathbb{R}^\Omega = U_1 \oplus \cdots \oplus U_m$. For $1 \leqslant i \leqslant m$, let $Q_i$ be the orthogonal projector onto $U_i$. Let $M$ be a symmetric matrix which commutes with $Q_i$ for $1 \leqslant i \leqslant m$. Let the eigenspaces of $M$ be $V_1, \ldots, V_r$. Let the non-zero subspaces among the intersections $U_i \cap V_j$ be $W_1, \ldots, W_t$. Then*

*(i)* $\mathbb{R}^\Omega = W_1 \oplus \cdots \oplus W_t$;

*(ii)* *if* $k \neq l$ *then* $W_k \perp W_l$;

*(iii) for $1 \leqslant k \leqslant t$, the space $W_k$ is contained in an eigenspace of $M$;*

*(iv) for $1 \leqslant k \leqslant t$, the orthogonal projector onto $W_k$ is a polynomial in $Q_1, \ldots, Q_m$ and $M$.*

**Proof** Part (iii) is immediate from the definition of the $W_k$. So is part (ii), because $U_{i_1} \perp U_{i_2}$ if $i_1 \neq i_2$ and $V_{j_1} \perp V_{j_2}$ if $j_1 \neq j_2$ so the two spaces $U_{i_1} \cap V_{j_1}$ and $U_{i_2} \cap V_{j_2}$ are either orthogonal or equal.

For $1 \leqslant j \leqslant r$, let $P_j$ be the projector onto $V_j$. Since $P_j$ is a polynomial in $M$, it commutes with $Q_i$, so $U_i$ is geometrically orthogonal to $V_j$. By Lemma 2.2, the orthogonal projector onto $U_i \cap V_j$ is $Q_iP_j$, which is a polynomial in $Q_i$ and $M$. This proves (iv).

Finally, $Q_iP_j = O$ precisely when $U_i \cap P_j = \{0\}$, so the sum of the orthogonal projectors onto the spaces $W_k$ is the sum of the non-zero products $Q_iP_j$, which is equal to the sum of all the products $Q_iP_j$. But

$$\sum_i \sum_j Q_iP_j = \left(\sum_i Q_i\right)\left(\sum_j P_j\right) = I^2 = I,$$

so Lemma 2.3 shows that $\mathbb{R}^\Omega = W_1 \oplus \cdots \oplus W_t$, proving (i). ■

**Corollary 2.5** *If $M$ and $N$ are commuting symmetric matrices in $\mathbb{R}^{\Omega\times\Omega}$ then $\mathbb{R}^\Omega$ is the direct sum of the non-zero intersections of the eigenspaces of $M$ and $N$. Moreover, these spaces are mutually orthogonal and their orthogonal projectors are polynomials in $M$ and $N$.*

**Proof** Apply Lemma 2.4 to $M$ and the eigenspaces of $N$. ■

If $W$ is contained in an eigenspace of a matrix $M$, I shall call it a *sub-eigenspace* of $M$.