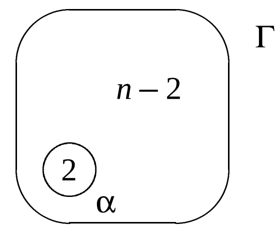
1.4 Some special association schemes

We have already met the trivial, group divisible and rectangular association schemes.

1.4.1 Triangular association schemes

Let Ω consist of all 2-subsets from an n-set Γ, so that $|\Omega| = n(n-1)/2$. For α in Ω, put



$$\begin{aligned}\mathcal{C}_1(\alpha) &= \{\beta \in \Omega : |\alpha \cap \beta| = 1\} \\ \mathcal{C}_2(\alpha) &= \{\beta \in \Omega : \alpha \cap \beta = \emptyset\}.\end{aligned}$$

Then $a_1 = 2(n-2)$ and

$$a_2 = {}^{n-2}\mathrm{C}_2 = (n-2)(n-3)/2.$$

Let ω, η, ζ and θ be distinct elements of Γ. If $\alpha = \{\omega, \eta\}$ and $\beta = \{\omega, \zeta\}$ then

$$\mathcal{C}_1(\alpha) \cap \mathcal{C}_1(\beta) = \{\{\omega, \gamma\} : \gamma \in \Gamma,\ \gamma \neq \omega,\ \eta,\ \zeta\} \cup \{\{\eta, \zeta\}\},$$

which has size $(n-3)+1 = n-2$. On the other hand, if $\alpha = \{\omega, \eta\}$ and $\beta = \{\zeta, \theta\}$ then

$$\mathcal{C}_1(\alpha) \cap \mathcal{C}_1(\beta) = \{\{\omega, \zeta\}, \{\omega, \theta\}, \{\eta, \zeta\}, \{\eta, \theta\}\},$$

which has size 4.

This is called the *triangular* association scheme $\mathrm{T}(n)$.

The labelling of the vertices of the Petersen graph in Figure 1.9 shows that the association scheme in Example 1.4 is $\mathrm{T}(5)$ (with the names of the classes interchanged—this does not matter).

1.4.2 Johnson schemes

More generally, let Ω consist of all m-subsets of an n-set Γ, where $1 \leqslant m \leqslant n/2$. For $i = 0, 1, \ldots, m$, let α and β be i-th associates if

$$|\alpha \cap \beta| = m - i,$$

so that smaller values of i give as i-th associates subsets which have larger overlap.

It is clear that each element has a_i i-th associates, where

$$a_i = {}^{m}\mathrm{C}_{m-i} \times {}^{n-m}\mathrm{C}_i$$

(see Figure 1.10).

We shall show later that this is an association scheme. It is called the *Johnson* scheme $\mathrm{J}(n,m)$. In particular, $\mathrm{J}(n,1) = \underline{\underline{n}}$ and $\mathrm{J}(n,2) = \mathrm{T}(n)$.

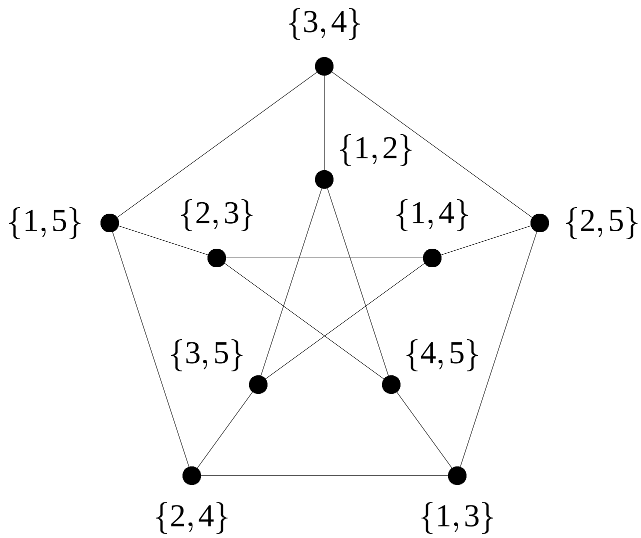


Figure 1.9: The Petersen graph as $T(5)$.

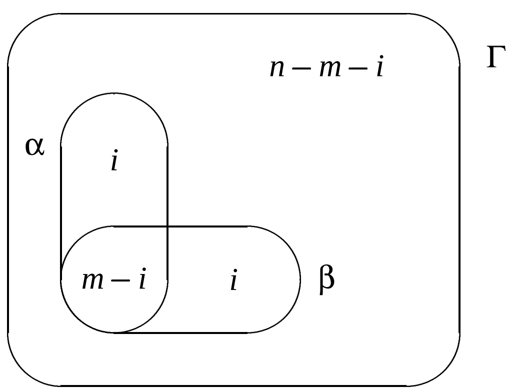


Figure 1.10: i-th associates in the Johnson scheme

1.4.3 Hamming schemes

Let Γ be an n-set and let $\Omega = \Gamma^m$. For α and β in Ω, let α and β be i-th associates if α and β differ in exactly i positions, where $0 \leqslant i \leqslant m$. Evidently, every element of Ω has a_i i-th associates, where

$$a_i = {}^m\mathrm{C}_i \times (n-1)^i.$$

We shall show later that this is an association scheme. It is called the *Hamming* scheme $\mathrm{H}(m,n)$.

The cube scheme in Example 1.5 is $\mathrm{H}(3,2)$, as the labelling in Figure 1.11 shows.

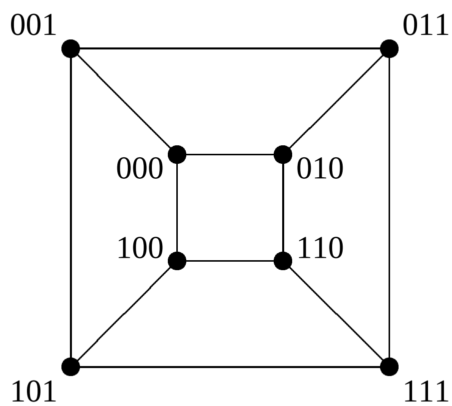


Figure 1.11: The cube labelled to show it as $\mathrm{H}(3,2)$

1.4.4 Distance-regular graphs

Recall that a *path of length l* between vertices α and β of a graph is a sequence of l edges $e_1, e_2, \ldots, e_l$ such that $\alpha \in e_1$, $\beta \in e_l$ and $e_i \cap e_{i+1} \neq \emptyset$ for $i = 1, \ldots, l-1$. The graph is *connected* if every pair of vertices is joined by a path.

The only strongly regular graphs which are not connected are disjoint unions of complete graphs of the same size. The graph formed by the red edges of the cube in Example 1.5 is such a graph; so is the graph formed by the black edges of the cube.

In a connected graph, the *distance* between two vertices is the length of a shortest path joining them. In particular, if $\{\alpha, \beta\}$ is an edge then the distance between α and β is 1, while the distance from α to itself is 0. The *diameter* of a connected graph is the maximum distance between any pair of its vertices.

All connected strongly regular graphs have diameter 2. The yellow edges of the cube form a connected graph of diameter 3.

In any connected graph $\mathcal{G}$ it is natural to defined subsets $\mathcal{G}_i$ of its vertex-set Ω by

$$\mathcal{G}_i = \{(\alpha,\beta) \in \Omega \times \Omega : \text{the distance between } \alpha \text{ and } \beta \text{ is } i\},$$

so that

$$\mathcal{G}_i(\alpha) = \{\beta \in \Omega : \text{the distance between } \alpha \text{ and } \beta \text{ is } i\}.$$

Definition Let $\mathcal{G}$ be a connected graph with diameter s and vertex-set Ω. Then $\mathcal{G}$ is *distance-regular* if $\mathcal{G}_0$, $\mathcal{G}_1$, ..., $\mathcal{G}_s$ form an association scheme on Ω.

In a distance-regular graph $\mathcal{G}$, there are integers a_i such that $|\mathcal{G}_i(\alpha)| = a_i$ for all α. In any connected graph, if $\beta \in \mathcal{G}_i(\alpha)$ then

$$\mathcal{G}_1(\beta) \subseteq \mathcal{G}_{i-1}(\alpha) \cup \mathcal{G}_i(\alpha) \cup \mathcal{G}_{i+1}(\alpha).$$

See Figure 1.12. Hence in a distance-regular graph $p^i_{j1} = 0$ if $i \neq 0$ and $j \notin \{i-1, i, i+1\}$. Put $b_i = p^i_{i-1,1}$ and $c_i = p^i_{i+1,1}$. It is interesting that the constancy of the a_i, b_i and c_i is enough to guarantee that the graph is distance-regular.

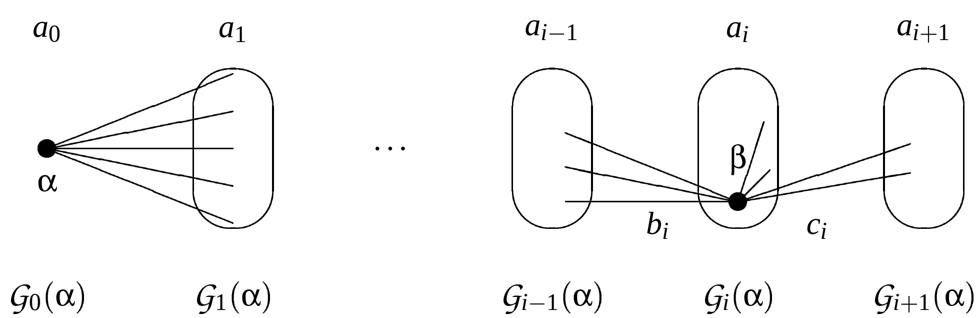


Figure 1.12: Edges through α and β in a distance-regular graph

Theorem 1.4 *Let Ω be the vertex-set of a connected graph $\mathcal{G}$ of diameter s. If there are integers a_i, b_i, c_i for $1 \leqslant i \leqslant s$ such that, for all α in Ω and all i in $\{1,\dots,s\}$,*

(a) $|\mathcal{G}_i(\alpha)| = a_i$;

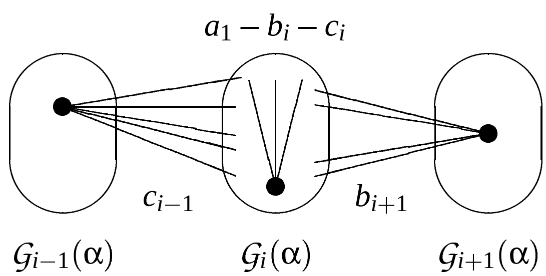
(b) if $\beta \in \mathcal{G}_i(\alpha)$ *then* $|\mathcal{G}_{i-1}(\alpha) \cap \mathcal{G}_1(\beta)| = b_i$ *and* $|\mathcal{G}_{i+1}(\alpha) \cap \mathcal{G}_1(\beta)| = c_i$

(so c_s must be 0) then $\mathcal{G}$ is distance-regular.

Proof It is clear that conditions (i) (diagonal subset) and (ii) (symmetry) in the definition of association scheme are satisfied. It remains to prove that A_iA_j is a linear combination of the A_k, where A_i is the adjacency matrix of the subset $\mathcal{G}_i$.

Let $i \in \{1,\dots,s-1\}$. Then

- every element in $\mathcal{G}_{i-1}(\alpha)$ is joined to exactly c_{i-1} elements of $\mathcal{G}_i(\alpha)$,
- every element in $\mathcal{G}_i(\alpha)$ is joined to exactly $a_1 - b_i - c_i$ elements of $\mathcal{G}_i(\alpha)$, and
- every element in $\mathcal{G}_{i+1}(\alpha)$ is joined to exactly b_{i+1} elements of $\mathcal{G}_i(\alpha)$.



Therefore

$$A_iA_1 = c_{i-1}A_{i-1} + (a_1 - b_i - c_i)A_i + b_{i+1}A_{i+1}.$$

Using the facts that $A_0 = I$ and all the b_i are non-zero (because the graph is connected), induction shows that, for $1 \leqslant i \leqslant s$,

- A_i is a polynomial in A_1 of degree i, and
- A_1^i is a linear combination of $I, A_1, \dots, A_i$.

Hence if $i, j \in \{0, \dots, s\}$ then A_iA_j is also a polynomial in A_1.

Applying a similar argument to $\mathcal{G}_s(\alpha)$ gives

$$A_sA_1 = c_{s-1}A_{s-1} + (a_1 - b_s)A_s$$

so A_1 satisfies a polynomial of degree $s+1$. Thus each product A_iA_j can be expressed as a polynomial in A_1 of degree at most s, so it is a linear combination of $A_0, A_1, \dots, A_s$. Hence condition (iii) is satisfied. ■

Example 1.5 revisited The yellow edges of the cube form a distance-regular graph with diameter 3 and the following parameters:

$$\begin{array}{lll} a_1 = 3 & a_2 = 3 & a_3 = 1 \\ b_1 = 1 & b_2 = 2 & b_3 = 3 \\ c_1 = 2 & c_2 = 1 & c_3 = 0. \end{array}$$ ■

In the Johnson scheme, draw an edge between α and ω if $|\alpha\cap\omega| = m-1$. Now suppose that $\beta \in \mathcal{C}_i(\alpha)$, so that $|\alpha\cap\beta| = m-i$, as in Figure 1.10. If $\eta \in \mathcal{C}_{i-1}(\alpha)\cap\mathcal{C}_1(\beta)$ then η is obtained from β by replacing one of the i elements in $\beta\setminus\alpha$ by one of the i elements in $\alpha\setminus\beta$, so β is joined to i^2 elements in $\mathcal{C}_{i-1}(\alpha)$. Similarly, if $\theta \in \mathcal{C}_{i+1}(\alpha)\cap\mathcal{C}_1(\beta)$ then θ is obtained from β by replacing one of the $m-i$ elements in $\alpha\cap\beta$ by one of the $n-m-i$ elements in $\Omega\setminus(\alpha\cup\beta)$, so β is joined to $(m-i)(n-m-i)$ elements in $\mathcal{C}_{i+1}(\alpha)$. Hence $\mathcal{C}_1$ defines a distance-regular graph, and so the Johnson scheme is indeed an association scheme.

Similarly, $\mathcal{C}_1$ in the Hamming scheme defines a distance-regular graph, so this is also a genuine association scheme.

(Warning: literature devoted to distance-regular graphs usually has a_i to denote what I am calling $a_1-b_i-c_i$.)

1.4.5 Cyclic association schemes

Cyclic association schemes are important in their own right, but this subsection also develops two other important ideas. One is the expression of the adjacency matrices as linear combinations of other, simpler, matrices whose products we already know, as in Example 1.6. The second is the introduction of algebras, which are central to the theory of association schemes.

Let Ω be $\mathbb{Z}_n$, the integers modulo n, considered as a group under addition.

(The reader who is familiar with group theory will realise that, throughout this subsection, the integers modulo n may be replaced by any finite Abelian group, so long as some of the detailed statements are suitably modified.)

We can define multiplication on F^Ω by

$$\chi_\alpha\chi_\beta = \chi_{\alpha+\beta} \qquad \text{for } \alpha \text{ and } \beta \text{ in } \Omega,$$

extended to the whole of F^Ω by

$$\left(\sum_{\alpha\in\Omega}\lambda_\alpha\chi_\alpha\right)\left(\sum_{\beta\in\Omega}\mu_\beta\chi_\beta\right) = \sum_{\alpha\in\Omega}\sum_{\beta\in\Omega}\lambda_\alpha\mu_\beta\chi_{\alpha+\beta}.$$

Thus, if f and g are in F^Ω then

$$(fg)(\omega) = \sum_{\alpha\in\Omega} f(\alpha)g(\omega-\alpha)$$

so this multiplication is sometimes called *convolution*. It is associative (because addition in $\mathbb{Z}_n$ is), is distributive over vector addition, and commutes with scalar multiplication, so it turns F^Ω into an *algebra*, called the *group algebra* of $\mathbb{Z}_n$, written $F\mathbb{Z}_n$.

Now let

$$\mathcal{D}_\omega = \{(\alpha,\beta) \in \Omega \times \Omega : \beta - \alpha = \omega\}$$

and let $M_\omega = A_{\mathcal{D}_\omega}$, so that the (α,β)-entry of M_ω is equal to 1 if $\alpha = \beta - \omega$ and to 0 otherwise. Then

$$M_\gamma M_\delta = M_{\gamma+\delta}$$

for γ and δ in Ω, and

$$M_\omega \chi_\gamma = \chi_{\gamma-\omega}$$

for ω and γ in Ω. Matrices of the form $\sum_{\omega\in\Omega} \lambda_\omega M_\omega$ for λ_ω in F are called *circulant* matrices.

Example 1.7 For $\mathbb{Z}_5$,

$$M_4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

and $M_1 = M_4'$. Circulant matrices are patterned in diagonal stripes. The general circulant matrix for $\mathbb{Z}_5$ is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} \lambda & \mu & \nu & \rho & \sigma \\ \sigma & \lambda & \mu & \nu & \rho \\ \rho & \sigma & \lambda & \mu & \nu \\ \nu & \rho & \sigma & \lambda & \mu \\ \mu & \nu & \rho & \sigma & \lambda \end{array}\right] \end{array}. \quad \blacksquare$$

Let $\mathcal{M}$ be the set of circulant matrices. It forms a vector space over F under matrix addition and scalar multiplication. It is also closed under matrix multiplication, which is associative, is distributive over matrix addition, and commutes with scalar multiplication. So $\mathcal{M}$ is also an algebra. The map

$$\varphi: F\mathbb{Z}_n \to \mathcal{M}$$

defined by

$$\varphi\left(\sum_{\omega\in\Omega} \lambda_\omega \chi_\omega\right) = \sum_{\omega\in\Omega} \lambda_\omega M_\omega$$

is a bijection which preserves the three operations (addition, scalar multiplication, multiplication) so it is an *algebra isomorphism*.

Notation If $\Delta \subseteq \mathbb{Z}_n$, write $-\Delta$ for $\{-\delta : \delta \in \Delta\}$ and, if $\gamma \in \Delta$, write $\gamma + \Delta$ for $\{\gamma + \delta : \delta \in \Delta\}$.

Definition A partition of $\mathbb{Z}_n$ into sets $\Delta_0, \Delta_1, \ldots, \Delta_s$ is a *blueprint* if

(i) $\Delta_0 = \{0\}$;

(ii) for $i = 1, \ldots, s$, if $\omega \in \Delta_i$ then $-\omega \in \Delta_i$ (that is, $\Delta_i = -\Delta_i$);

(iii) there are integers q_{ij}^k such that if $\beta \in \Delta_k$ then there are precisely q_{ij}^k elements α in Δ_i such that $\beta - \alpha \in \Delta_j$.

Condition (iii) says that there are exactly q_{ij}^k ordered pairs in $\Delta_i \times \Delta_j$ the sum of whose elements is equal to any given element in Δ_k. It implies that

$$\chi_{\Delta_i}\chi_{\Delta_j} = \sum_k q_{ij}^k \chi_{\Delta_k}. \tag{1.2}$$

Note that

$$\chi_{\Delta_0} = \sum_{\omega \in \Delta_0} \chi_\omega = \chi_0;$$

the first '0' is an element of the labelling set $\{0, 1, \ldots, s\}$ and the final '0' is the zero element of $\mathbb{Z}_n$.

Suppose that $\Delta_0, \Delta_1, \ldots, \Delta_s$ do form a blueprint for $\mathbb{Z}_n$. Put $\mathcal{C}_i = \bigcup_{\omega \in \Delta_i} \mathcal{D}_\omega$, so that

$$\mathcal{C}_i(\alpha) = \{\beta \in \Omega : \alpha - \beta \in \Delta_i\} = \alpha - \Delta_i = \alpha + \Delta_i$$

by condition (ii). In particular, $|\mathcal{C}_i(\alpha)| = |\Delta_i|$. Moreover, each $\mathcal{C}_i$ is symmetric, and $\mathcal{C}_0 = \mathrm{Diag}(\Omega)$. Now the adjacency matrix A_i of $\mathcal{C}_i$ is given by

$$A_i = \sum_{\omega \in \Delta_i} M_\omega = \sum_{\omega \in \Delta_i} \varphi(\chi_\omega) = \varphi\left(\sum_{\omega \in \Delta_i} \chi_\omega\right) = \varphi(\chi_{\Delta_i})$$

so Equation (1.2) shows that

$$\begin{aligned} A_i A_j &= \varphi(\chi_{\Delta_i})\varphi(\chi_{\Delta_j}) = \varphi(\chi_{\Delta_i}\chi_{\Delta_j}) \\ &= \varphi(\sum_k q_{ij}^k \chi_{\Delta_k}) = \sum_k q_{ij}^k \varphi(\chi_{\Delta_k}) \\ &= \sum_k q_{ij}^k A_k \end{aligned}$$

and hence the $\mathcal{C}_i$ form an association scheme on Ω. It is called a *cyclic* association scheme.

Thus a partition of the smaller set Ω with the right properties (a blueprint) leads to a partition of the larger set $\Omega \times \Omega$ with the right properties (an association scheme). The former is much easier to check.

Example 1.8 In $\mathbb{Z}_{13}$, put $\Delta_0 = \{0\}$, $\Delta_1 = \{1,3,4,-4,-3,-1\}$ and $\Delta_2 = \Omega \setminus \Delta_0 \setminus \Delta_1$. We calculate the sums $\alpha+\beta$ for $\alpha, \beta \in \Delta_1$.

	1	3	4	−4	−3	−1
1	2	4	5	−3	−2	0
3	4	6	−6	−1	0	2
4	5	−6	−5	0	1	3
−4	−3	−1	0	5	6	−5
−3	−2	0	1	6	−6	−4
−1	0	2	3	−5	−4	−2

In the body of the table

0	occurs	6 times
1, 3, 4, −4, −3, −1	each occur	2 times
2, 5, 6, −6, −5, −2	each occur	3 times

so $\chi_{\Delta_1}\chi_{\Delta_1} = 6\chi_{\Delta_0} + 2\chi_{\Delta_1} + 3\chi_{\Delta_2}$.

As usual for $s = 2$, there is nothing more to check, for

$$\begin{aligned}\chi_{\Delta_1}\chi_{\Delta_2} &= \chi_{\Delta_1}(\chi_\Omega - \chi_0 - \chi_{\Delta_1})\\ &= 6\chi_\Omega - \chi_{\Delta_1} - \chi_{\Delta_1}\chi_{\Delta_1}\\ &= 6(\chi_0 + \chi_{\Delta_1} + \chi_{\Delta_2}) - \chi_{\Delta_1} - (6\chi_0 + 2\chi_{\Delta_1} + 3\chi_{\Delta_2})\\ &= 3\chi_{\Delta_1} + 3\chi_{\Delta_2}\end{aligned}$$

and

$$\begin{aligned}\chi_{\Delta_2}\chi_{\Delta_2} &= \chi_{\Delta_2}(\chi_\Omega - \chi_0 - \chi_{\Delta_1})\\ &= 6\chi_\Omega - \chi_{\Delta_2} - \chi_{\Delta_1}\chi_{\Delta_2}\\ &= 6(\chi_0 + \chi_{\Delta_1} + \chi_{\Delta_2}) - \chi_{\Delta_2} - (3\chi_{\Delta_1} + 3\chi_{\Delta_2})\\ &= 6\chi_0 + 3\chi_{\Delta_1} + 2\chi_{\Delta_2}.\end{aligned}$$

So Δ_0, Δ_1, Δ_2 form a blueprint for $\mathbb{Z}_{13}$. ■

For any n, the sets $\{0\}$, $\{\pm 1\}$, $\{\pm 2\}$, … form a blueprint for $\mathbb{Z}_n$. The corresponding cyclic association scheme is the same as the one derived from the circuit C_n with n vertices, which is a distance-regular graph. Call this association scheme $\textcircled{n}$.