1.3 Matrices

Given a field F, the set F^Ω of functions from Ω to F forms a vector space. Addition is defined pointwise: for f and g in F^Ω,

$$(f+g)(\omega) = f(\omega)+g(\omega) \qquad \text{for } \omega \text{ in } \Omega.$$

Scalar multiplication is also defined pointwise: for λ in F and f in F^Ω,

$$(\lambda f)(\omega) = \lambda(f(\omega)) \qquad \text{for } \omega \text{ in } \Omega.$$

For ω in Ω, let χ_ω be the characteristic function of ω; that is

$$\begin{aligned} \chi_\omega(\omega) &= 1 \\ \chi_\omega(\alpha) &= 0 \qquad \text{for } \alpha \text{ in } \Omega \text{ with } \alpha \neq \omega. \end{aligned}$$

Then $f = \sum_{\omega\in\Omega} f(\omega)\chi_\omega$ for all f in F^Ω, and so $\{\chi_\omega : \omega \in \Omega\}$ forms a natural basis of F^Ω: hence $\dim F^\Omega = |\Omega|$.

For any subset Δ of Ω it is also convenient to define χ_Δ in F^Ω by

$$\chi_\Delta(\omega) = \begin{cases} 1 & \text{if } \omega \in \Delta \\ 0 & \text{if } \omega \notin \Delta. \end{cases}$$

so that $\chi_\Delta = \sum_{\omega\in\Delta}\chi_\omega$.

Given two finite sets Γ and Δ, we can form their product

$$\Gamma\times\Delta = \{(\gamma,\delta) : \gamma\in\Gamma,\ \delta\in\Delta\},$$

which may be thought of as a rectangle, as in Figure 1.7. Applying the preceding

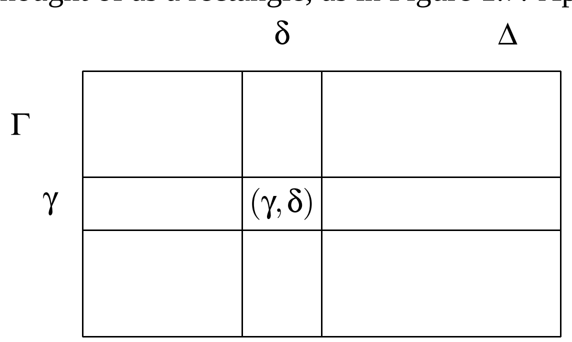

Figure 1.7: The product set $\Gamma\times\Delta$

ideas to $\Gamma\times\Delta$, we obtain the vector space $F^{\Gamma\times\Delta}$ of all functions from $\Gamma\times\Delta$ to F. It has dimension $|\Gamma|\times|\Delta|$. If M is such a function, we usually write $M(\gamma,\delta)$

rather than $M((\gamma,\delta))$. In fact, M is just a matrix, with its rows labelled by Γ and its columns labelled by Δ. So long as we retain this labelling, it does not matter in what order we write the rows and columns.

In fact, what most people regard as an $m\times n$ matrix over F is just a matrix in $F^{\Gamma\times\Delta}$ with $\Gamma=\{1,\ldots,m\}$ and $\Delta=\{1,\ldots,n\}$. The usual convention is that row 1 appears first, etc., so that order *does* matter but labelling is not needed. Here I use the opposite convention (order does not matter, but labelling is needed) because usually the elements of Γ and Δ are not integers.

Some examples of matrices with such labelled rows and columns were given in Example 1.5.

If $M\in F^{\Gamma\times\Delta}$ and $\Gamma=\Delta$ then we say that M is *square*. This is stronger than merely having the same number of rows as of columns. If we need to emphasize the set, we say that M is square on Δ.

The *transpose* of M is the matrix M' in $F^{\Delta\times\Gamma}$ defined by

$$M'(\delta,\gamma)=M(\gamma,\delta)\qquad\text{for }\delta\text{ in }\Delta\text{ and }\gamma\text{ in }\Gamma.$$

The matrix M in $F^{\Delta\times\Delta}$ is *symmetric* if $M'=M$. There are three special symmetric matrices in $F^{\Delta\times\Delta}$:

$$\begin{aligned}
I_\Delta(\delta_1,\delta_2) &= \begin{cases}1 & \text{if } \delta_1=\delta_2\\ 0 & \text{otherwise;}\end{cases}\\
J_\Delta(\delta_1,\delta_2) &= 1 \qquad \text{for all } \delta_1,\ \delta_2 \text{ in } \Delta;\\
O_\Delta(\delta_1,\delta_2) &= 0 \qquad \text{for all } \delta_1,\ \delta_2 \text{ in } \Delta.
\end{aligned}$$

Moreover, if $f\in F^\Delta$ we define the symmetric matrix $\mathrm{diag}(f)$ in $F^{\Delta\times\Delta}$ by

$$\mathrm{diag}(f)(\delta_1,\delta_2)=\begin{cases}f(\delta_1) & \text{if }\delta_1=\delta_2\\ 0 & \text{otherwise.}\end{cases}$$

Matrix multiplication is possible when the labelling sets are compatible. If $M_1\in F^{\Gamma\times\Delta}$ and $M_2\in F^{\Delta\times\Phi}$ then M_1M_2 is the matrix in $F^{\Gamma\times\Phi}$ defined by

$$(M_1M_2)(\gamma,\phi)=\sum_{\delta\in\Delta}M_1(\gamma,\delta)M_2(\delta,\phi).$$

All the usual results about matrix multiplication hold. In particular, matrix multiplication is associative, and $(M_1M_2)'=M_2'M_1'$.

Similarly, if $M\in F^{\Gamma\times\Delta}$ then M defines a linear transformation from F^Δ to F^Γ by

$$f\mapsto Mf$$

where

$$(Mf)(\gamma)=\sum_{\delta\in\Delta}M(\gamma,\delta)f(\delta)\qquad\text{for }\gamma\in\Gamma.$$

If Φ is any subset of $\Gamma \times \Delta$ then its characteristic function χ_Φ satisfies

$$\chi_\Phi(\gamma,\delta) = \begin{cases} 1 & \text{if } (\gamma,\delta) \in \Phi \\ 0 & \text{otherwise.} \end{cases}$$

In the special case that $\Gamma = \Delta = \Omega$ we call χ_Φ the *adjacency matrix* of Φ and write it A_Φ. In particular,

$$\begin{aligned} A_{\Omega\times\Omega} &= J_\Omega \\ A_\emptyset &= O_\Omega \\ A_{\mathrm{Diag}(\Omega)} &= I_\Omega. \end{aligned}$$

For an association scheme with classes $\mathcal{C}_0$, $\mathcal{C}_1$, …, $\mathcal{C}_s$, we write A_i for the adjacency matrix $A_{\mathcal{C}_i}$. Thus the (α,β)-entry of A_i is equal to 1 if α and β are i-th associates; otherwise it is equal to 0.

Condition (iii) in the definition of association scheme has a particularly nice consequence for multiplication of the adjacency matrices: it says that

$$A_iA_j = \sum_{k=0}^{s} p_{ij}^k A_k. \tag{1.1}$$

For suppose that $(\alpha,\beta) \in \mathcal{C}_k$. Then the (α,β)-entry of the right-hand side of Equation (1.1) is equal to p_{ij}^k, while the (α,β)-entry of the left-hand side is equal to

$$\begin{aligned} \left(A_iA_j\right)(\alpha,\beta) &= \sum_{\gamma\in\Omega} A_i(\alpha,\gamma)A_j(\gamma,\beta) \\ &= \left|\left\{\gamma : (\alpha,\gamma) \in \mathcal{C}_i \text{ and } (\gamma,\beta) \in \mathcal{C}_j\right\}\right| \\ &= p_{ij}^k \end{aligned}$$

because the product $A_i(\alpha,\gamma)A_j(\gamma,\beta)$ is zero unless $(\alpha,\gamma) \in \mathcal{C}_i$ and $(\gamma,\beta) \in \mathcal{C}_j$, in which case it is 1.

This leads to a definition of association schemes in terms of the adjacency matrices.

Definition (Third definition of association scheme) An *association scheme* with s associate classes on a finite set Ω is a set of matrices A_0, A_1, …, A_s in $\mathbb{R}^{\Omega\times\Omega}$, all of whose entries are equal to 0 or 1, such that

(i)″ $A_0 = I_\Omega$;

(ii)″ A_i is symmetric for $i = 1, \ldots, s$;

(iii)″ for all i, j in $\{1,\ldots,s\}$, the product A_iA_j is a linear combination of A_0, A_1, …, A_s;

(iv)$''$ none of the A_i is equal to O_Ω, and $\sum_{i=0}^{s} A_i = J_\Omega$.

Notice that we do not need to specify that the coefficients in (iii)$''$ be integers: *every* entry in A_iA_j is a non-negative integer if all the entries in A_i and A_j are 0 or 1. Condition (iv)$''$ is the analogue of the sets $\mathcal{C}_0$, ..., $\mathcal{C}_s$ forming a partition of $\Omega \times \Omega$.

Since A_i is symmetric with entries in $\{0, 1\}$, the diagonal entries of A_i^2 are the row-sums of A_i. Condition (iii)$''$ implies that A_i^2 has a constant element, say a_i, on its diagonal. Therefore every row and every column of A_i contains a_i entries equal to 1. Hence $A_iJ_\Omega = J_\Omega A_i = a_iJ_\Omega$. Moreover, $A_0A_i = A_iA_0 = A_i$. Thus condition (iii)$''$ can be checked by, for each i in $\{1, \ldots, s\}$, verifying that A_i has constant row-sums and that, for all but one value of j in $\{1, \ldots, s\}$, the product A_iA_j is a linear combination of A_0, ..., A_s. In fact, the first check is superfluous, because it is a byproduct of checking A_i^2.

Example 1.6 Let Λ be a Latin square of size n: an $n \times n$ array filled with n letters in such a way that each letter occurs once in each row and once in each column. A Latin square of size 4 is shown in Figure 1.8(a).

Let Ω be the set of n^2 cells in the array. For α, β in Ω with $\alpha \neq \beta$ let α and β be first associates if α and β are in the same row or are in the same column or have the same letter, and let α and β be second associates otherwise. Figure 1.8(b) shows one element α and marks all its first associates as β.

A	*B*	*C*	*D*
D	*A*	*B*	*C*
C	*D*	*A*	*B*
B	*C*	*D*	*A*

(a) A Latin square

β		β	
β			β
α	β	β	β
β	β		

(b) An element α and all its first associates β

Figure 1.8: A association scheme of Latin-square type

We need to check that all the nine products A_iA_j are linear combinations of A_0, A_1 and A_2. Five products involve A_0, which is I, so there is nothing to check. (Here and elsewhere we omit the suffix from I, J etc. when the set involved is clear from the context.) I claim that only the product A_1^2 needs to be checked.

	A_0	A_1	A_2
A_0	$\surd$	$\surd$	$\surd$
A_1	$\surd$	?	
A_2	$\surd$		

To check A_1^2, we need a simple expression for A_1. Let R be the adjacency matrix of the subset

$$\{(\alpha,\beta) : \alpha \text{ and } \beta \text{ are in the same row}\},$$

and let C and L be the adjacency matrices of the similarly-defined subsets for columns and letters. Then

$$A_1 = R + C + L - 3I,$$

because the elements of $\mathrm{Diag}(\Omega)$ are counted in each of R, C, and L and need to be removed. Moreover, $A_2 = J - A_1 - I$. These adjacency matrices have constant row-sums: $a_1 = 3(n-1)$ and $a_2 = n^2 - 3(n-1) - 1 = (n-2)(n-1)$.

Now

$$\begin{aligned} R^2(\alpha,\beta) &= \sum_\gamma R(\alpha,\gamma)R(\gamma,\beta) \\ &= |\{\gamma : \gamma \text{ is the same row as } \alpha \text{ and } \beta\}| \\ &= \begin{cases} n & \text{if } \alpha \text{ and } \beta \text{ are in the same row} \\ 0 & \text{otherwise} \end{cases} \\ &= nR(\alpha,\beta) \end{aligned}$$

so $R^2 = nR$. Similarly $C^2 = nC$ and $L^2 = nL$. Also

$$\begin{aligned} RC(\alpha,\beta) &= \sum_\gamma R(\alpha,\gamma)C(\gamma,\beta) \\ &= |\{\gamma : \gamma \text{ is in the same row as } \alpha \text{ and the same column as } \beta\}| \\ &= 1 \end{aligned}$$

so $RC = J$. Similarly $CR = RL = LR = CL = LC = J$.

Hence $A_1^2 = n(R+C+L) + 6J - 6(R+C+L) + 9I$, which is a linear combination of A_1, J and I, hence of A_1, A_2 and A_0.

Now let us verify my claim that no further products need to be evaluated. We do not need to check A_1A_2, because

$$A_1A_2 = A_1(J - A_1 - I) = a_1J - A_1^2 - A_1.$$

Neither do we need to check A_2A_1, because

$$A_2A_1 = (J - A_1 - I)A_1 = a_1J - A_1^2 - A_1.$$

Finally, we do not need to check A_2^2 either, because

$$A_2^2 = A_2(J - A_1 - I) = a_2J - A_2A_1 - A_2.$$

This association scheme is said to be of *Latin-square type* $\mathrm{L}(3,n)$. ■

It is no accident that $A_1A_2 = A_2A_1$ in Example 1.6.

Lemma 1.2 *If $A_0, A_1, \ldots, A_s$ are the adjacency matrices of an association scheme then $A_iA_j = A_jA_i$ for all i, j in $\{0,1,\ldots s\}$.*

Proof

$$\begin{aligned}
A_jA_i &= A_j'A_i', && \text{because the adjacency matrices are symmetric,}\\
&= (A_iA_j)'\\
&= \left(\sum_k p_{ij}^k A_k\right)'\\
&= \sum_k p_{ij}^k A_k'\\
&= \sum_k p_{ij}^k A_k, && \text{because the adjacency matrices are symmetric,}\\
&= A_iA_j. \quad \blacksquare
\end{aligned}$$

As we saw in Example 1.6, there is very little to check when there are only two associate classes.

Lemma 1.3 *Let A be a symmetric matrix in $\mathbb{R}^{\Omega\times\Omega}$ with zeros on the diagonal and all entries in $\{0,1\}$. Suppose that $A \neq O$ and $A \neq J - I$. Then $\{I, A, J - A - I\}$ is an association scheme on Ω if and only if A^2 is a linear combination of I, A and J.*

Consideration of the α-th row of A_iA_j sheds new light on the graph way of looking at association schemes. Stand at the vertex α. Take a step along an edge coloured i (if $i = 0$ this means stay still). Then take a step along an edge coloured j. Where can you get to? If β is joined to α by a k-coloured edge, then you can get to β in this way if $p_{ij}^k \neq 0$. In fact, there are exactly p_{ij}^k such two-step ways of getting to β from α.