4.3 Orthogonal block structures

Definition An *orthogonal block structure* on a set Ω is a set $\mathcal{F}$ of pairwise orthogonal uniform partitions of Ω which is closed under $\wedge$ and $\vee$ (in particular, $\mathcal{F}$ contains U and E).

Thus, in Example 4.1, the set $\{E$, rows, columns, rectangles, $U\}$ is an orthogonal block structure. So is the set $\{E$, rows, columns, letters, $U\}$ in Example 4.2.

Note that, if all the other conditions for an orthogonal block structure are met, then orthogonality between partitions F and G is easy to check: whenever $\{\alpha,\beta\}$ is contained in an F-class and $\{\beta,\gamma\}$ is contained in a G-class then there must be some element δ such that $\{\alpha,\delta\}$ is contained in a G-class and $\{\delta,\gamma\}$ is contained in an F-class. That is, wherever can be reached in the two-colour graph for $F \vee G$ by a red edge followed by a blue edge can also be reached by a blue edge followed by a red edge.

Theorem 4.8 *Let $\mathcal{F}$ be an orthogonal block structure on Ω. For F in $\mathcal{F}$, define the subset $\mathcal{C}_F$ of $\Omega \times \Omega$ by*

$$(\alpha,\beta) \in \mathcal{C}_F \quad \text{if} \quad F = \bigwedge \{G \in \mathcal{F} : \alpha \text{ and } \beta \text{ are in the same } G\text{-class}\}.$$

Then $\{\mathcal{C}_F : F \in \mathcal{F},\ \mathcal{C}_F \neq \emptyset\}$ forms an association scheme on Ω with valencies a_F, where

$$a_F = \sum_{G \in \mathcal{F}} \mu(G,F) k_G.$$

Proof The non-empty $\mathcal{C}_F$ do form a partition of $\Omega \times \Omega$, because $\mathcal{F}$ is closed under $\wedge$. They are symmetric. The equality partition E is in $\mathcal{F}$ and $\mathcal{C}_E = \mathrm{Diag}(\Omega)$.

Let A_F be the adjacency matrix for $\mathcal{C}_F$. Then

$$\alpha \text{ and } \beta \text{ are in the same } F\text{-class} \iff \text{there is some } G \preccurlyeq F \text{ with } (\alpha,\beta) \in \mathcal{C}_G,$$

so

$$R_F = \sum_{G \preccurlyeq F} A_G = \sum_{G \in \mathcal{F}} \zeta(G,F) A_G = \sum_{G \in \mathcal{F}} \zeta'(F,G) A_G. \tag{4.3}$$

This is true for all F in $\mathcal{F}$. The inverse of the matrix ζ' is μ', so Equation (4.3) can be inverted (this is called *Möbius inversion*) to give

$$A_F = \sum_{G \in \mathcal{F}} \mu'(F,G) R_G = \sum_{G \in \mathcal{F}} \mu(G,F) R_G. \tag{4.4}$$

Taking row sums of Equation (4.4) gives

$$a_F = \sum_{G \in \mathcal{F}} \mu'(F,G)k_G = \sum_{G \in \mathcal{F}} \mu(G,F)k_G. \tag{4.5}$$

As usual, let $\mathcal{A}$ be the subspace of $\mathbb{R}^{\Omega \times \Omega}$ spanned by $\{A_F : F \in \mathcal{F}\}$. We must show that $\mathcal{A}$ is closed under multiplication. Equations (4.3) and (4.4) show that $\mathcal{A} = \operatorname{span}\{R_F : F \in \mathcal{F}\}$. All the partitions are uniform, so Proposition 4.1 shows that $\mathcal{A} = \operatorname{span}\{P_F : F \in \mathcal{F}\}$. The partitions are pairwise orthogonal, and $\mathcal{F}$ is closed under $\vee$, so we can apply Lemma 4.4 (i) and deduce that $\operatorname{span}\{P_F : F \in \mathcal{F}\}$ is closed under multiplication. ■

Theorem 4.9 *Let $\mathcal{F}$ be an orthogonal block structure on Ω. For F in $\mathcal{F}$, put*

$$W_F = V_F \cap \left(\sum_{G \succ F} V_G \right)^{\perp}.$$

Then the non-zero spaces W_F, for F in $\mathcal{F}$, are the strata for the association scheme. Their projectors S_F satisfy

$$S_F = \sum_{G \in \mathcal{F}} \mu(F,G)P_G,$$

and their dimensions d_F satisfy

$$d_F = \sum_{G \in \mathcal{F}} \mu(F,G)n_G.$$

Proof Theorem 4.7 shows that, for all F in $\mathcal{F}$,

$$V_F = \bigoplus_{G \succcurlyeq F} W_G$$

and the summands are orthogonal. So if S_F is the projector onto W_F then

$$P_F = \sum_{G \succcurlyeq F} S_G = \sum_{G \in \mathcal{F}} \zeta(F,G)S_G. \tag{4.6}$$

Möbius inversion gives

$$S_F = \sum_{G \in \mathcal{F}} \mu(F,G)P_G. \tag{4.7}$$

Thus each S_F is in the Bose-Mesner algebra $\mathcal{A}$ of the association scheme. But each S_F is idempotent, so Lemma 2.7 shows that each S_F is a sum of zero or more stratum projectors. If $F \neq G$ then $W_F \perp W_G$, so S_F and S_G cannot contain any stratum projectors in common. Thus no linear combination of $\{S_F : F \in \mathcal{F}\}$ projects onto any non-zero proper subspace of W_G, for any G in $\mathcal{F}$. But Equations (4.6) and (4.7) show that $\mathcal{A} = \operatorname{span}\{S_F : F \in \mathcal{F}\}$, so the non-zero spaces W_F must be precisely the strata.

Taking the trace of both sides of Equation (4.7) gives

$$d_F = \sum_{G \in \mathcal{F}} \mu(F,G) n_G, \tag{4.8}$$

because $\operatorname{tr} P_G = \dim V_G = n_G$. ■

The character table of the association scheme follows immediately from the work done so far. From Equations (4.4) and (4.6) and Proposition 4.1 we have

$$\begin{aligned} A_F &= \sum_G \mu'(F,G) R_G \\ &= \sum_G \mu'(F,G) k_G P_G \\ &= \sum_G \mu'(F,G) k_G \sum_H \zeta(G,H) S_H, \end{aligned}$$

so

$$C = \mu' \operatorname{diag}(k) \zeta. \tag{4.9}$$

Inversion then gives

$$D = C^{-1} = \mu \operatorname{diag}(k)^{-1} \zeta'. \tag{4.10}$$

We can check that this agrees with the results found in Chapter 2. From Equations (4.5) and (4.8) we obtain the (symmetric) diagonal matrices of valencies and dimensions as

$$\operatorname{diag}(a) = \mu' \operatorname{diag}(k) = \operatorname{diag}(k)\mu$$

and

$$\operatorname{diag}(d) = \mu \operatorname{diag}(n).$$

Applying Corollary 2.13 to Equation (4.9) gives

$$\begin{aligned} D &= \frac{1}{|\Omega|} \operatorname{diag}(d) C' \operatorname{diag}(a)^{-1} \\ &= \frac{1}{|\Omega|} \mu \operatorname{diag}(n) \zeta' \operatorname{diag}(k) \mu \mu^{-1} \operatorname{diag}(k)^{-1} \\ &= \frac{1}{|\Omega|} \mu \operatorname{diag}(n) \zeta' \\ &= \mu \operatorname{diag}(k)^{-1} \zeta' \end{aligned}$$

because $n_F k_F = |\Omega|$ for all F.

Although the existence of the Möbius function is useful for proving general results, such as Theorems 4.8 and 4.9, it is rarely used in explicit calculations. The adjacency matrices and their valencies can be calculated recursively, using Equations (4.4) and (4.5), starting at the bottom of the Hasse diagram. Likewise, the stratum projectors and the dimensions of the strata are calculated recursively, using Equations (4.7) and (4.8), starting at the top of the Hasse diagram.

Example 4.1 revisited Write R, C and B respectively for R_{rows}, R_{columns} and $R_{\text{rectangles}}$. We have $k_E = 1$, $k_{\text{rows}} = 4$, $k_{\text{columns}} = 2$, $k_{\text{rectangles}} = 8$ and $k_U = 24$. Working from the bottom of the Hasse diagram in Figure 4.1, we obtain

$$\begin{array}{lcllcr} A_E & = & I & a_E & = & 1 \\ A_{\text{rows}} & = & R-I & a_{\text{rows}} & = & 3 \\ A_{\text{columns}} & = & C-I & a_{\text{columns}} & = & 1 \\ A_{\text{rectangles}} & = & B-R-C+I & a_{\text{rectangles}} & = & 3 \\ A_U & = & J-B & a_U & = & 16. \end{array}$$

To obtain the strata, we start at the top of the Hasse diagram. For the dimensions we use the fact that $n_U = 1$, $n_{\text{rectangles}} = 3$, $n_{\text{columns}} = 12$, $n_{\text{rows}} = 6$ and $n_E = 24$:

$$\begin{array}{lcllcl} S_U & = & \frac{1}{24}J & d_U & = & 1 \\ S_{\text{rectangles}} & = & \frac{1}{8}B - \frac{1}{24}J & d_{\text{rectangles}} & = & 2 \\ S_{\text{columns}} & = & \frac{1}{2}C - \frac{1}{8}B & d_{\text{columns}} & = & 9 \\ S_{\text{rows}} & = & \frac{1}{4}R - \frac{1}{8}B & d_{\text{rows}} & = & 3 \\ S_E & = & I - \frac{1}{4}R - \frac{1}{2}C + \frac{1}{8}B & d_E & = & 9. \end{array}$$

The character table is now either calculated directly by expressing each adjacency matrix above in terms of the stratum projectors, or by using Equation (4.9). If we keep the elements of $\mathcal{F}$ in the order

$$E \quad \text{rows} \quad \text{columns} \quad \text{rectangles} \quad U,$$

we obtain

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 0 & 4 & 4 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 3 & -1 & 3 & 3 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & -3 & -1 & 3 & 3 \\ 0 & 0 & 0 & -8 & 16 \end{bmatrix}.$$

Notice that the entries 1 come in the *first* row, while the valencies come in the *last* column.

Likewise we calculate

$$D = \frac{1}{24} \begin{bmatrix} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 24 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{24} \begin{bmatrix} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 24 & 0 & 0 & 0 & 0 \\ 6 & 6 & 0 & 0 & 0 \\ 12 & 0 & 12 & 0 & 0 \\ 3 & 3 & 3 & 3 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{24} \begin{bmatrix} 9 & -3 & -9 & 3 & 0 \\ 3 & 3 & -3 & -3 & 0 \\ 9 & -3 & 9 & -3 & 0 \\ 2 & 2 & 2 & 2 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Here the dimensions come in the first column, while the 1s come in the last row. ■

This example displays a difficulty which was alluded to at the end of Section 2.2. In the association scheme of an orthogonal block structure, the associate classes and the strata are both naturally labelled by $\mathcal{F}$, so we normally have $\mathcal{K} = \mathcal{E} = \mathcal{F}$. However, the diagonal associate class, usually called $\mathcal{C}_0$, is here $\mathcal{C}_E$, while the all-1s stratum, usually called W_0, is here W_U. In other words, the special associate class and the special stratum correspond to the two different trivial partitions. That is why the special entries in the matrices C and D are not all in the first row and column.

On the other hand, the foregoing example also demonstrates an advantage that orthogonal block structures have over association schemes in general. There are explicit, straightforward formulae for the strata and character table, so there is no need to resort to any of the techniques in Section 2.4.

If any of the $\mathcal{C}_F$ is empty then $\mathcal{K}$ is a proper subset of $\mathcal{F}$. Theorem 2.6 shows that $|\mathcal{E}| = |\mathcal{K}|$, so $\mathcal{E}$ is also a proper subset of $\mathcal{F}$; in fact, the number of F such that $W_F = 0$ is the same as the number of F such that $\mathcal{C}_F = \emptyset$. However, the partitions which give zero subspaces are not usually the same as those which give empty subsets of $\Omega \times \Omega$.

Example 4.2 revisited Although this association scheme is defined by a single Latin square, it is *not* of the Latin square type $\mathrm{L}(3,n)$ described in Example 1.6: the former first associates have been separated into those in the same row, those in the same column and those in the same letter. If we write R, C and L for R_{rows}, R_{columns} and R_{letters}, we obtain

$$\begin{array}{llllll}
A_E & = & I & a_E & = & 1 \\
A_{\text{rows}} & = & R-I & a_{\text{rows}} & = & n-1 \\
A_{\text{columns}} & = & C-I & a_{\text{columns}} & = & n-1 \\
A_{\text{letters}} & = & L-I & a_{\text{letters}} & = & n-1 \\
A_U & = & J-R-C-L+2I & a_U & = & (n-1)(n-2)
\end{array}$$

and

$$\begin{array}{llllll}
S_U & = & \frac{1}{n^2}J & d_U & = & 1 \\
S_{\text{rows}} & = & \frac{1}{n}R-\frac{1}{n^2}J & d_{\text{rows}} & = & n-1 \\
S_{\text{columns}} & = & \frac{1}{n}C-\frac{1}{n^2}J & d_{\text{columns}} & = & n-1 \\
S_{\text{letters}} & = & \frac{1}{n}L-\frac{1}{n^2}J & d_{\text{letters}} & = & n-1 \\
S_E & = & I-\frac{1}{n}R-\frac{1}{n}C-\frac{1}{n}L+\frac{2}{n^2}J & d_E & = & (n-1)(n-2).
\end{array}$$

If $n = 2$ then $\mathcal{C}_U = \emptyset$ and $W_E = 0$. ■

More generally, if we have $c-2$ mutually orthogonal $n \times n$ Latin squares then we can create an orthogonal block structure with c non-trivial partitions of a set of size n^2 into n classes of size n. It is also an orthogonal array. If $c = n+1$ then $\mathcal{C}_U = \emptyset$ and $W_E = 0$.