Chapter 4

Families of Partitions

4.1 A partial order on partitions

We have seen that the group-divisible association scheme is defined by a partition of Ω into blocks of equal size. A particularly tidy basis for the Bose-Mesner algebra $\mathcal{A}$ in this case is $\{I, B, J\}$, where B is the adjacency matrix for the relation "is in the same block as". The main object of this chapter is to give a single general construction for association schemes defined by families of partitions of Ω.

Let F be a partition of Ω into n_F subsets, which I shall call F-classes. (I use the letter F because statisticians call partitions *factors*.) Define the relation matrix R_F in $\mathbb{R}^{\Omega} \times \Omega$ by

$$R_F(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same } F\text{-class} \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

just as B was defined in Section 3.1. Then, as in Section 3.4, define the subspace V_F of $\mathbb{R}^{\Omega}$ by

$$V_F = \left\{ v \in \mathbb{R}^{\Omega} : v(\alpha) = v(\beta) \text{ whenever } \alpha \text{ and } \beta \text{ are in the same } F\text{-class} \right\}.$$

Then $\dim V_F = n_F$. Let P_F be the orthogonal projector onto V_F. Then $(P_F v)(\alpha)$ is equal to the average of the values $v(\beta)$ for β in the same F-class as α.

Definition The partition F is *uniform* if all classes of F have the same size.

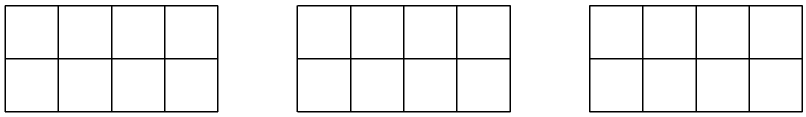
(Unfortunately, there is no consensus about what to call such a partition. The words *uniform*, *regular*, *balanced* and *proper* are all in use.)

Proposition 4.1 *If F is uniform with classes of size k_F then $P_F = k_F^{-1} R_F$.*

So much for a single partition. Now we consider what happens when we have two partitions.

If F and G are both partitions of Ω, write $F \preccurlyeq G$ if every F-class is contained in a G-class; write $F \prec G$ if $F \preccurlyeq G$ and $F \neq G$. We may pronounce $F \preccurlyeq G$ as "F is finer than G" or "G is coarser than F". (Statisticians often say "F is nested in G".)

Example 4.1 In the association scheme $\underline{\underline{3}}/(\underline{\underline{2}} \times \underline{\underline{4}})$ the set Ω consists of 24 elements, divided into three rectangles as follows.



The set is partitioned in three ways—into rows, columns and rectangles. Identifying the names of the partitions with the names of their classes, we have rows $\prec$ rectangles and columns $\prec$ rectangles. ■

Lemma 4.2 *If $F \preccurlyeq G$ then $V_G \leqslant V_F$.*

Proof Any function which is constant on G-classes must be constant on F-classes too if each F-class is contained in a G-class. ■

There are two special, but trivial, partitions on every set with more than one element. The *universal* partition U consists of a single class containing the whole of Ω (the *universe*). At the other extreme, the *equality* partition E has as its classes all the singleton subsets of Ω; in other words, α and β are in the same E-class if and only if $\alpha = \beta$. Note that $E \preccurlyeq F \preccurlyeq U$ for all partitions F.

The relation $\preccurlyeq$ satisfies:

(i) (**reflexivity**) for every partition F, $F \preccurlyeq F$;

(ii) (**anti-symmetry**) if $F \preccurlyeq G$ and $G \preccurlyeq F$ then $F = G$;

(iii) (**transitivity**) if $F \preccurlyeq G$ and $G \preccurlyeq H$ then $F \preccurlyeq H$.

This means that $\preccurlyeq$ is a *partial order*. Partial orders are often shown on *Hasse diagrams*: there is a dot for each element (partition in this case); if $F \prec G$ then F is drawn below G and is joined to G by a line or sequence of lines, all going generally upwards.

Example 4.1 revisited The Hasse diagram for this example is in Figure 4.1. ■

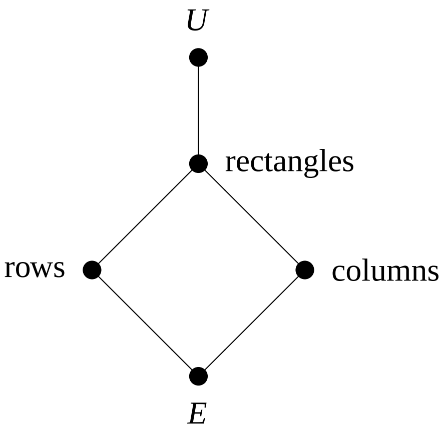


Figure 4.1: Hasse diagram for Example 4.1

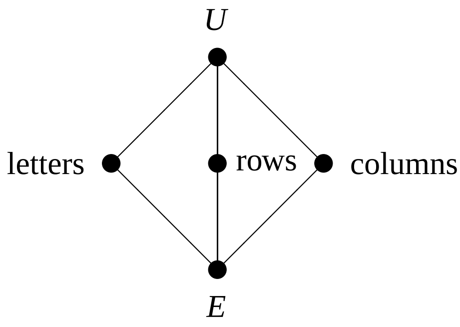


Figure 4.2: Hasse diagram for Example 4.2

Example 4.2 The set of n^2 cells in a $n \times n$ Latin square has three non-trivial partitions—into rows, columns and letters. The Hasse diagram in shown in Figure 4.2. ■

A set with a partial order on it is often called a *poset*. Every poset has a zeta function and a Möbius function, which I shall now define for our particular partial order.

Let $\mathcal{F}$ be any set of partitions of Ω. Define ζ in $\mathbb{R}^{\mathcal{F}\times\mathcal{F}}$ by

$$\zeta(F,G) = \begin{cases} 1 & \text{if } F \preccurlyeq G \\ 0 & \text{otherwise.} \end{cases}$$

In fact, the partial order $\preccurlyeq$ is often considered to be the subset $\{(F,G) : F \preccurlyeq G\}$ of $\mathcal{F} \times \mathcal{F}$. From this viewpoint, ζ is just the adjacency matrix of $\preccurlyeq$.

The elements of $\mathcal{F}$ can be written in an order such that F comes before G if $F \prec G$. Then ζ is an upper triangular matrix with 1s on the diagonal, so it has an inverse matrix μ which is also upper triangular with integer entries and 1s on the diagonal. The matrix μ is called the *Möbius* function of the poset $(\mathcal{F}, \preccurlyeq)$.

Example 4.1 revisited Here

$$\zeta = \begin{array}{c} \\ E \\ \text{rows} \\ \text{columns} \\ \text{rectangles} \\ U \end{array} \begin{array}{c} \begin{array}{ccccc} E & \text{rows} & \text{columns} & \text{rectangles} & U \end{array} \\ \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

so

$$\mu = \begin{array}{r} \\ E \\ \text{rows} \\ \text{columns} \\ \text{rectangles} \\ U \end{array} \begin{array}{c} \begin{array}{ccccc} E & \text{rows} & \text{columns} & \text{rectangles} & U \end{array} \\ \left[\begin{array}{ccccc} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \qquad \blacksquare$$

Example 4.2 revisited For the Latin square,

$$\zeta = \begin{array}{r} \\ E \\ \text{rows} \\ \text{columns} \\ \text{letters} \\ U \end{array} \begin{array}{c} \begin{array}{ccccc} E & \text{rows} & \text{columns} & \text{letters} & U \end{array} \\ \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

and so

$$\mu = \begin{array}{r} \\ E \\ \text{rows} \\ \text{columns} \\ \text{letters} \\ U \end{array} \begin{array}{c} \begin{array}{ccccc} E & \text{rows} & \text{columns} & \text{letters} & U \end{array} \\ \left[\begin{array}{ccccc} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \qquad \blacksquare$$

If F and G are partitions, let $F \wedge G$ be the partition whose classes are the non-empty intersections of F-classes with G-classes. Then

(i) $F \wedge G \preccurlyeq F$ and $F \wedge G \preccurlyeq G$, and

(ii) if $H \preccurlyeq F$ and $H \preccurlyeq G$ then $H \preccurlyeq F \wedge G$,

so $F \wedge G$ is the *infimum* (or *greatest lower bound*) of F and G. Dually, the *supremum* (or *least upper bound*) $F \vee G$ of F and G is defined by

(i) $F \preccurlyeq F \vee G$ and $G \preccurlyeq F \vee G$, and

(ii) if $F \preccurlyeq H$ and $G \preccurlyeq H$ then $F \vee G \preccurlyeq H$.

To construct $F \vee G$, draw a coloured graph whose vertex set is Ω. There is a red edge between α and β if α and β are in the same F-class, and a blue edge if α and β are in the same G-class. Then the classes of $F \vee G$ are the connected components of the red-and-blue graph.

Example 4.1 revisited In the example with rectangular arrays, we have rows $\wedge$ columns $= E$ and rows $\vee$ columns $=$ rectangles.

Example 4.2 revisited In the Latin square, the supremum of any two of the non-trivial partitions is U and the infimum of any two is E.

Of course, if $F \preccurlyeq G$ then $F \wedge G = F$ and $F \vee G = G$. Moreover, just as the empty sum is 0 and the empty product is 1, we make the convention that the empty infimum is U and the empty supremum is E.

Lemma 4.3 *If F and G are partitions of Ω, then $V_F \cap V_G = V_{F\vee G}$.*

Proof Let v be a vector in $\mathbb{R}^\Omega$. Then

$$\begin{aligned} v \in V_F \cap V_G \quad &\iff \quad \text{v is constant on each component of the blue graph and v is constant on each component of the red graph} \\ &\iff \quad \text{v is constant on each component of the red-and-blue graph} \\ &\iff \quad v \in V_{F\vee G}. \quad \blacksquare \end{aligned}$$

4.2 Orthogonal partitions

Definition Let F and G be partitions of Ω. Then F is *orthogonal* to G if V_F is geometrically orthogonal to V_G, that is, if $P_F P_G = P_G P_F$.

Note that this implies that F is orthogonal to G if $F \preccurlyeq G$, because $P_F P_G = P_G P_F = P_G$ in that case.

Lemma 4.4 *If F is orthogonal to G, then*

(i) $P_F P_G = P_{F\vee G}$;

(ii) $V_F \cap (V_{F\vee G})^\perp$ *is orthogonal to* V_G.

Proof These both follow immediately from Lemma 2.2, using the fact that $V_F \cap V_G = V_{F\vee G}$. $\blacksquare$

Corollary 4.5 *Partitions F and G are orthogonal to each other if and only if,*

(i) within each class of $F \vee G$, each F-class meets every G-class, and

(ii) for each element ω *of* Ω,

$$\frac{|F\text{-class containing }\omega|}{|F \vee G\text{-class containing }\omega|} = \frac{|F \wedge G\text{-class containing }\omega|}{|G\text{-class containing }\omega|}.$$

Corollary 4.6 *If* F *is orthogonal to* G *and if* F, G *and* $F \vee G$ *are all uniform then* $F \wedge G$ *is also uniform and*

$$k_{F\wedge G}k_{F\vee G} = k_F k_G.$$

Example 4.1 revisited Here rows are orthogonal to columns even though no row meets every column. ■

Example 4.2 revisited As in Example 1.6, write R, C and L for R_{rows}, R_{columns} and R_{letters}. Proposition 4.1 shows that $P_{\mathrm{rows}} = n^{-1}R$, $P_{\mathrm{columns}} = n^{-1}C$, $P_{\mathrm{letters}} = n^{-1}L$ and $P_U = n^{-2}J$. We saw in Example 1.6 that $RC = CR = RL = LR = CL = LR = J$. Thus the partitions into rows, columns and letters are pairwise orthogonal. ■

Theorem 4.7 *Let* $\mathcal{F}$ *be a set of pairwise orthogonal partitions of* Ω *which is closed under* $\vee$. *For* F *in* $\mathcal{F}$, *put*

$$W_F = V_F \cap \left(\sum_{G \succ F} V_G\right)^{\perp}.$$

Then

(i) the spaces W_F *and* W_G *are orthogonal to each other whenever* F *and* G *are different partitions in* $\mathcal{F}$;

(ii) for each F *in* $\mathcal{F}$,

$$V_F = \bigoplus_{G \succcurlyeq F} W_G.$$

Proof (i) If $F \neq G$ then $F \vee G$ must be different from at least one of F and G. Suppose that $F \vee G \neq F$. Then $F \vee G \succ F$ and $F \vee G \in \mathcal{F}$ so

$$W_F \leqslant V_F \cap V_{F\vee G}^{\perp},$$

while $W_G \leqslant V_G$. Lemma 4.4 (ii) shows that $V_F \cap V_{F\vee G}^{\perp}$ is orthogonal to V_G, because F is orthogonal to G. Hence W_F is orthogonal to W_G.

(ii) Since the spaces W_G, for $G \succcurlyeq F$, are pairwise orthogonal, their vector space sum is direct, so it suffices to prove that

$$V_F = \sum_{G \succcurlyeq F} W_G. \tag{4.2}$$

We do this by induction. If there is no G such that $G \succ F$ then $W_F = V_F$ and Equation (4.2) holds. The definition of W_F shows that

$$V_F = W_F + \sum_{H \succ F} V_H.$$

If the inductive hypothesis is true for every H with $H \succ F$ then

$$\sum_{H \succ F} V_H = \sum_{H \succ F} \sum_{G \succcurlyeq H} W_G.$$

If $G \succcurlyeq H \succ F$ then $G \succ F$, and if $G \succ F$ then $G \succcurlyeq G \succ F$ so

$$\sum_{H \succ F} \sum_{G \succcurlyeq H} W_G = \sum_{G \succ F} W_G.$$

Thus

$$V_F = W_F + \sum_{G \succ F} W_G = \sum_{G \succcurlyeq F} W_G. \quad \blacksquare$$