## 3.7 Cyclic designs

Let $\Theta = \mathbb{Z}_t$. For $\Phi \subseteq \Theta$, a *translate* of $\Phi$ is a set of the form

$$\Phi + \theta = \{\phi + \theta : \phi \in \Phi\}$$

for some $\theta$ in $\Theta$. Of course, $\Phi$ is a translate of itself.

It is possible to have $\Phi + \theta_1 = \Phi + \theta_2$ even when $\theta_1 \neq \theta_2$. Then $\Phi + (\theta_1 - \theta_2) = \Phi$. Let $l$ be the number of distinct translates of $\Phi$: we shall abuse group-theoretic terminology slightly and refer to $l$ as the *index* of $\Phi$. Then $\Phi + (l \bmod t) = \Phi$. Moreover, $l$ is the smallest positive integer with this property, and $l$ divides $t$ (for if not, the remainder on dividing $t$ by $l$ is a smaller positive number $l'$ with $\Phi + (l' \bmod t) = \Phi$).

**Definition** An incomplete-block design with treatment set $\mathbb{Z}_t$ is a *thin cyclic design* if there is some subset $\Phi$ of $\mathbb{Z}_t$ such that the blocks are all the distinct translates of $\Phi$: the design is said to be *generated* by $\Phi$. An incomplete-block design is a *cyclic* design if its blocks can be partitioned into sets of blocks such that each set is a thin cyclic design.

**Example 3.13** Let $\Phi = \{0,1,3\} \subset \mathbb{Z}_8$. This has index 8, so it generates the following thin cyclic design.

$$\{0,1,3\},\ \{1,2,4\},\ \{2,3,5\},\ \{3,4,6\},\ \{4,5,7\},\ \{5,6,0\},\ \{6,7,1\},\ \{7,0,2\}. \quad \blacksquare$$

**Example 3.14** Here is a cyclic design for $\mathbb{Z}_6$ which is not thin.

$$\{0,1,4\},\ \{1,2,5\},\ \{2,3,0\},\ \{3,4,1\},\ \{4,5,2\},\ \{5,0,3\},\ \{0,2,4\},\ \{1,3,5\}.$$

The index of $\{0,1,4\}$ is 6 and the index of $\{0,2,4\}$ is 2. $\blacksquare$

**Theorem 3.13** *Let $\Phi \subset \mathbb{Z}_t$ and let $l$ be the index of $\Phi$. For $\theta$ in $\mathbb{Z}_t$, let*

$$m_\theta(\Phi) = |\{(\phi_1,\phi_2) \in \Phi \times \Phi : \phi_1 - \phi_2 = \theta\}|,$$

*so that*

$$\chi_\Phi \chi_{-\Phi} = \sum_{\theta \in \Theta} m_\theta(\Phi)\chi_\theta.$$

*Then, in the thin cyclic design generated by $\Phi$,*

$$\Lambda(0,\theta) = m_\theta(\Phi) \times \frac{l}{t}$$

*and*

$$\Lambda(\eta,\zeta) = \Lambda(0,\zeta - \eta). \tag{3.7}$$

**Proof** Treatments 0 and $\theta$ concur in the translate $\Phi+\psi$ if and only if there are $\phi_1$, $\phi_2$ in $\Phi$ such that $\phi_1+\psi=\theta$ and $\phi_2+\psi=0$, that is $\psi=-\phi_2$ and $\theta=\phi_1-\phi_2$. If $l=t$ then $\Lambda(0,\theta)=m_\theta(\Phi)$. In general, the family of sets $\Phi$, $\Phi+1$, ..., $\Phi+t-1$ consists of $t/l$ copies of the $l$ distinct translates $\Phi$, $\Phi+1$, ..., $\Phi+l-1$, so the concurrence in the thin design is $(l/t)m_\theta(\Phi)$.

Moreover, treatments 0 and $\theta$ concur in $\Phi+\psi$ if and only if treatments $\eta$ and $\eta+\theta$ concur in $\Phi+\psi+\eta$, so $\Lambda(0,\theta)=\Lambda(\eta,\eta+\theta)$. ■

**Corollary 3.14** *Every cyclic design is partially balanced with respect to the cyclic association scheme on $\mathbb{Z}_t$ defined by the blueprint $\{0\}$, $\{\pm1\}$, $\{\pm2\}$, .... (It* may *be partially balanced with respect to a cyclic association scheme with fewer associate classes.)*

**Proof** Since Equation (3.7) holds in each thin component of the design, it holds overall, and

$$\Lambda=\sum_{\theta\in\Theta}\Lambda(0,\theta)M_\theta,$$

where

$$M_\theta(\eta,\zeta)=\begin{cases}1 & \text{if } \zeta-\eta=\theta\\ 0 & \text{otherwise,}\end{cases}$$

as in Section 1.4.5. But $\Lambda$ is symmetric, so $\Lambda(0,-\theta)=\Lambda(-\theta,0)=\Lambda(0,\theta)$, by Equation (3.7). The adjacency matrices for the cyclic association scheme defined by the blueprint $\{0\}$, $\{\pm1\}$, $\{\pm2\}$ ... are $(M_\theta+M_{-\theta})$ if $2\theta\neq0$ and $M_\theta$ if $2\theta=0$, so $\Lambda$ is a linear combination of the adjacency matrices, and so the design is partially balanced with respect to this association scheme.

Suppose that $\Delta_0$, $\Delta_1$, ..., $\Delta_s$ is a blueprint for $\mathbb{Z}_t$ such that $\Lambda(0,\theta)$ is constant $\lambda_i$ for $\theta$ in $\Delta_i$. Putting $A_i=\sum_{\theta\in\Delta_i}M_\theta$ gives $\Lambda=\sum_i\lambda_iA_i$, and so the design is partially balanced with respect to the cyclic association scheme defined by the blueprint. ■

Now write $\lambda_\theta$ for $\Lambda(0,\theta)$.

**Technique 3.8** To calculate the concurrences in the thin design generated by $\Phi$, form the *table of differences* for $\Phi$. Try to find the coarsest blueprint such that $\lambda_\theta$ is constant on each set in the partition.

**Example 3.13 revisited** In $\mathbb{Z}_8$, the block $\{0,1,3\}$ gives the following table of differences.

| | 0 | 1 | 3 |
|---|---|---|---|
| 0 | 0 | 1 | 3 |
| 1 | 7 | 0 | 2 |
| 3 | 5 | 6 | 0 |

Therefore $\lambda_0 = 3$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = \lambda_6 = \lambda_7 = 1$ and $\lambda_4 = 0$. Hence the design is partially balanced for the association scheme defined by the blueprint $\{0\}$, $\{4\}$, $\{1,2,3,5,6,7\}$ (so this design is group divisible with groups $0,4 \parallel 1,5 \parallel, 2,6 \parallel 3,7$). ■

**Definition** A subset $\Phi$ of $\mathbb{Z}_t$ is a *perfect difference set* for $\mathbb{Z}_t$ if there are integers $r$, $\lambda$ such that

$$\chi_\Phi \chi_{-\Phi} = r\chi_0 + \lambda(\chi_{\mathbb{Z}_t} - \chi_0);$$

in other words, $m_\theta(\Phi) = \lambda$ for all $\theta$ with $\theta \neq 0$.

**Proposition 3.15** *The thin cyclic design generated by $\Phi$ is balanced if and only if $\Phi$ is a perfect difference set.*

**Example 3.5 revisited** The subset $\{1,2,4\}$ is a perfect difference set for $\mathbb{Z}_7$.

| | 1 | 2 | 4 |
|---|---|---|---|
| 1 | 0 | 1 | 3 |
| 2 | 6 | 0 | 2 |
| 4 | 4 | 5 | 0 |

Its table of differences contains every non-zero element of $\mathbb{Z}_7$ exactly once. ■

**Theorem 3.16** *The canonical efficiency factors of a cyclic design are*

$$1 - \frac{1}{rk} \sum_{\theta \in \mathbb{Z}_t} \lambda_\theta \eta^\theta$$

*for complex $t$-th roots of unity $\eta$ with $\eta \neq 1$.*

**Proof** Use Theorems 3.12 and 2.18. ■

**Technique 3.9** Let $\zeta = \exp\left(\frac{2\pi i}{t}\right)$. Then $\eta$ is a complex $t$-th root of unity if there is an integer $m$ such that $\eta = \zeta^m$. To calculate canonical efficiency factors of cyclic designs numerically, replace $\eta^\theta + \eta^{-\theta}$ by $2\cos\left(\frac{2\pi\theta m}{t}\right)$. To calculate the harmonic mean efficiency factor $A$ as an exact rational number, leave everything in powers of $\zeta$.

**Example 3.15** Consider the thin cyclic design generated by $\{0,1,3,7\}$ in $\mathbb{Z}_9$.

| | 0 | 1 | 3 | 7 |
|---|---|---|---|---|
| 0 | 0 | 1 | 3 | 7 |
| 1 | 8 | 0 | 2 | 6 |
| 3 | 6 | 7 | 0 | 4 |
| 7 | 2 | 3 | 5 | 0 |

Thus the eigenvalues of $\Lambda$ are

$$4+(\eta+\eta^{-1})+2(\eta^2+\eta^{-2})+2(\eta^3+\eta^{-3})+(\eta+\eta^{-4})$$

where $\eta^9=1$. If $\eta^3=1$ and $\eta\neq 1$ then $\eta+\eta^{-1}=-1$ (the cube roots of unity sum to zero) so the eigenvalue is

$$4-1-2+4-1=4;$$

otherwise it is

$$4+\eta^2+\eta^{-2}-2=2+\eta^2+\eta^{-2},$$

because the primitive ninth roots of unity sum to zero (because all the ninth roots do). Let $\zeta$ be a fixed primitive ninth root of unity, and put $x=\zeta+\zeta^{-1}$, $y=\zeta^2+\zeta^{-2}$ and $z=\zeta^4+\zeta^{-4}$. Then the canonical efficiency factors are

$$\frac{3}{4},\quad \frac{14-x}{16},\quad \frac{14-y}{16},\quad \frac{14-z}{16},$$

all with multiplicity 2.

Substituting $x=2\cos 40^o$, $y=2\cos 80^o$, $z=2\cos 160^o$ gives

$$0.7500,\quad 0.7792,\quad 0.8533\quad \text{and}\quad 0.9925$$

to 4 decimal places, and $A=0.8340$.

To do the exact calculation, we note first that $x+y+z=0$. Then

$$\begin{aligned} xy &= (\zeta+\zeta^{-1})(\zeta^2+\zeta^{-2}) \\ &= \zeta+\zeta^3+\zeta^{-3}+\zeta^{-1} \\ &= x-1, \end{aligned}$$

and similarly $yz=y-1$ and $zx=z-1$. Therefore $xy+yz+zx=x+y+z-3=-3$ and $xyz=(x-1)z=xz-z=z-1-z=-1$.

Now

$$\begin{aligned} &\frac{1}{14-x}+\frac{1}{14-y}+\frac{1}{14-x} \\ &\quad= \frac{(14-x)(14-y)+(14-x)(14-z)+(14-y)(14-z)}{(14-x)(14-y)(14-z)} \\ &\quad= \frac{3\cdot 14^2-28(x+y+z)+(xy+yz+zx)}{14^3-14^2(x+y+z)+14(xy+yz+zx)-xyz} \\ &\quad= \frac{3\cdot 14^2-3}{14^3-3\cdot 14+1}=\frac{195}{901}, \end{aligned}$$

so

$$4A^{-1} = \frac{4}{3} + \frac{16 \times 195}{901}$$

so

$$A^{-1} = \frac{1}{3} + \frac{4 \times 195}{901} = \frac{3241}{2703}$$

and

$$A = \frac{2703}{3241}. \quad \blacksquare$$