

Chapter 1

Association Schemes

1.1 Partitions

Association schemes are about relations between pairs of elements of a set Ω . In this book Ω will always be finite. Recall that $\Omega \times \Omega$ is the set of ordered pairs of elements of Ω ; that is,

$$\Omega \times \Omega = \{(\alpha, \beta) : \alpha \in \Omega, \beta \in \Omega\}.$$

I shall give three equivalent definitions of association scheme, in terms of partitions, graphs and matrices respectively. Each definition has advantages and disadvantages, depending on the context.

Recall that a *partition* of a set Δ is a set of non-empty subsets of Δ which are mutually disjoint and whose union is Δ .

Let C be any subset of $\Omega \times \Omega$. Its *dual subset* is C' , where

$$C' = \{(\beta, \alpha) : (\alpha, \beta) \in C\}.$$

We say that C is *symmetric* if $C = C'$. One special symmetric subset is the *diagonal* subset $\text{Diag}(\Omega)$ defined by

$$\text{Diag}(\Omega) = \{(\omega, \omega) : \omega \in \Omega\}.$$

Definition (First definition of association scheme) An *association scheme* with s associate classes on a finite set Ω is a partition of $\Omega \times \Omega$ into sets C_0, C_1, \dots, C_s (called *associate classes*) such that

- (i) $C_0 = \text{Diag}(\Omega)$;
- (ii) C_i is symmetric for $i = 1, \dots, s$;

(iii) for all i, j, k in $\{0, \dots, s\}$ there is an integer p_{ij}^k such that, for all (α, β) in C_k ,

$$|\{\gamma \in \Omega : (\alpha, \gamma) \in C_i \text{ and } (\gamma, \beta) \in C_j\}| = p_{ij}^k.$$

Note that the superscript k in p_{ij}^k does not signify a power.

Elements α and β of Ω are called i -th associates if $(\alpha, \beta) \in C_i$.

We can visualize Ω as a square array whose rows and columns are indexed by Ω , as in Figure 1.1. If the rows and columns are indexed in the same order, then the diagonal subset consists of the elements on the main diagonal, which are marked 0 in Figure 1.1. Condition (i) says that C_0 is precisely this diagonal subset. Condition (ii) says that every other associate class is symmetric about that diagonal: if the whole picture is reflected about that diagonal then the associate classes remain the same. For example, the set of elements marked * could form an associate class if symmetry were the only requirement. Condition (iii) is much harder to visualize for partitions, but is easier to interpret in the later two definitions.

Ω

	0	*			*		
	*	0					
			0				*
Ω				0			
	*				0		
						0	
			*				0

Figure 1.1: The elements of $\Omega \times \Omega$

Note that condition (ii) implies that $p_{ij}^0 = 0$ if $i \neq j$. Similarly, $p_{0j}^k = 0$ if $j \neq k$ and $p_{i0}^k = 0$ if $i \neq k$, while $p_{0j}^j = 1 = p_{i0}^i$. Condition (iii) implies that every element of Ω has p_{ii}^0 i -th associates, so that in fact the set of elements marked * in Figure 1.1 could not be an associate class. Write $a_i = p_{ii}^0$. This is called the *valency* of the i -th associate class. (Many authors use n_i to denote valency, but this conflicts with the very natural use of n_i in Chapters ?? and ??.)

The integers $|\Omega|$, s , a_i for $1 \leq i \leq s$ and p_{ij}^k for $1 \leq i, j, k \leq s$ are called the *parameters of the first kind*. Note that $p_{ij}^k = p_{ji}^k$.

Example 1.1 Let $\Delta_1 \cup \dots \cup \Delta_b$ be a partition of Ω into b subsets of size m , where $b \geq 2$ and $m \geq 2$. These subsets are traditionally called *groups*, even though they have nothing to do with the algebraic structure called a group. Let α and β be

- first associates if they are in the same group but $\alpha \neq \beta$;
- second associates if they are in different groups.

See Figure 1.2.

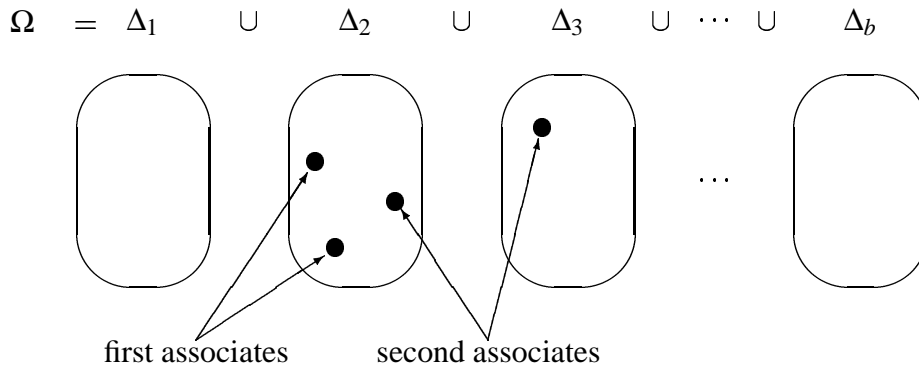


Figure 1.2: Partition of Ω in Example 1.1.

If $\omega \in \Omega$ then ω has $m - 1$ first associates and $(b - 1)m$ second associates.

If α and β are first associates then the number of γ which are specified associates of α and β are:

	first associate of β	second associate of β
first associate of α	$m - 2$	0
second associate of α	0	$(b - 1)m$

For example, those elements which are first associates of both α and β are the $m - 2$ other elements in the group which contains α and β . If α and β are second associates then the number of γ which are specified associates of α and β are:

	first associate of β	second associate of β
first associate of α	0	$m - 1$
second associate of α	$m - 1$	$(b - 2)m$

So this is an association scheme with $s = 2$, $a_1 = m - 1$, $a_2 = (b - 1)m$, and

$$\begin{array}{cccc}
 p_{11}^1 = m - 2 & p_{12}^1 = 0 & p_{11}^2 = 0 & p_{12}^2 = m - 1 \\
 p_{21}^1 = 0 & p_{22}^1 = (b - 1)m & p_{21}^2 = m - 1 & p_{22}^2 = (b - 2)m
 \end{array}$$

It is called the *group divisible* association scheme, denoted $\text{GD}(b, m)$ or $\underline{b}/\underline{m}$. (The name “group divisible” has stuck, because that is the name originally used by Bose

and Nair. But anyone who uses the word “group” in its algebraic sense is upset by this. It seems to me to be quite acceptable to call the scheme just “divisible”. Some authors tried to compromise by calling it “groop-divisible”, but the nonce word “groop” has not found wide approval.) ■

To save writing phrases like “first associates of α ” we introduce the notation $C_i(\alpha)$ for the set of i -th associates of α . That is

$$C_i(\alpha) = \{\beta \in \Omega : (\alpha, \beta) \in C_i\}.$$

Thus condition (iii) says that $|C_i(\alpha) \cap C_j(\beta)| = p_{ij}^k$ if $\beta \in C_k(\alpha)$.

Example 1.2 Let $|\Omega| = n$, let C_0 be the diagonal subset and let $C_1 = \{(\alpha, \beta) \in \Omega \times \Omega : \alpha \neq \beta\} = \Omega \times \Omega \setminus C_0$. This is the *trivial* association scheme—the only association scheme on Ω with only one associate class. It has $a_1 = n - 1$ and $p_{11}^1 = n - 2$. I shall denote it \underline{n} . ■

Example 1.3 Let Ω be an $n \times m$ rectangular array with $n \geq 2$ and $m \geq 2$, as in Figure 1.3. Note that this is a picture of Ω itself, not of $\Omega \times \Omega$! Put

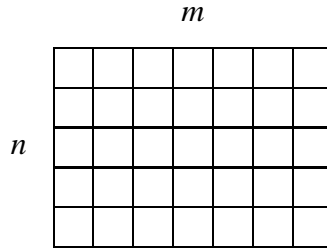


Figure 1.3: The set Ω in Example 1.3

$$\begin{aligned} C_1 &= \{(\alpha, \beta) : \alpha, \beta \text{ are in the same row but } \alpha \neq \beta\} \\ C_2 &= \{(\alpha, \beta) : \alpha, \beta \text{ are in the same column but } \alpha \neq \beta\} \\ C_3 &= \{(\alpha, \beta) : \alpha, \beta \text{ are in different rows and columns}\} \\ &= \Omega \times \Omega \setminus C_0 \setminus C_1 \setminus C_2. \end{aligned}$$

Then $a_1 = m - 1$, $a_2 = n - 1$ and $a_3 = (m - 1)(n - 1)$.

If $(\alpha, \beta) \in C_1$ then the number of γ which are specified associates of α and β are:

	$C_1(\beta)$	$C_2(\beta)$	$C_3(\beta)$
$C_1(\alpha)$	$m - 2$	0	0
$C_2(\alpha)$	0	0	$n - 1$
$C_3(\alpha)$	0	$n - 1$	$(n - 1)(m - 2)$

The entries in the above table are the p_{ij}^1 .

If $(\alpha, \beta) \in C_2$ then the number of γ which are specified associates of α and β are:

	$C_1(\beta)$	$C_2(\beta)$	$C_3(\beta)$
$C_1(\alpha)$	0	0	$m-1$
$C_2(\alpha)$	0	$n-2$	0
$C_3(\alpha)$	$m-1$	0	$(n-2)(m-1)$

The entries in the above table are the p_{ij}^2 .

Finally, if $(\alpha, \beta) \in C_3$ then the number of γ which are specified associates of α and β are:

	$C_1(\beta)$	$C_2(\beta)$	$C_3(\beta)$
$C_1(\alpha)$	0	1	$m-2$
$C_2(\alpha)$	1	0	$n-2$
$C_3(\alpha)$	$m-2$	$n-2$	$(n-2)(m-2)$

and the entries in the above table are the p_{ij}^3 .

This is the *rectangular* association scheme $R(n, m)$ or $\underline{n} \times \underline{m}$. It has three associate classes. ■

Lemma 1.1 (i) $\sum_{i=0}^s a_i = |\Omega|$;

(ii) for every i and k , $\sum_j p_{ij}^k = a_i$.

Proof (i) The set Ω is the disjoint union of $C_0(\alpha), C_1(\alpha), \dots, C_s(\alpha)$.

(ii) Given any (α, β) in C_k , the set $C_i(\alpha)$ is the disjoint union of the sets $C_i(\alpha) \cap C_j(\beta)$ for $j = 0, 1, \dots, s$. ■

Thus it is sufficient to check constancy of the a_i for all but one value of i , and, for each pair (i, k) , to check the constancy of p_{ij}^k for all but one value of j .

Thus construction of tables like those above is easier if we include a row for $C_0(\alpha)$ and a column for $C_0(\beta)$. Then the figures in the i -th row and column must sum to a_i , so we begin by putting these totals in the margins of the table, then calculate the easier entries (remembering that the table must be symmetric), then finish off by subtraction.

Example 1.4 Let Ω consist of the vertices of the Petersen graph, which is shown in Figure 1.4. Let C_1 consist of the edges of the graph and C_2 consist of the non-edges (that is, of those pairs of distinct vertices which are not joined by an edge).

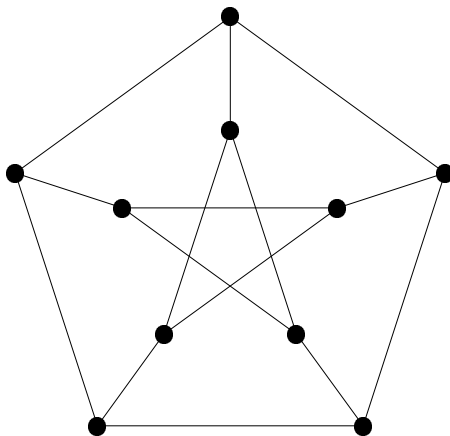


Figure 1.4: The Petersen graph

Inspection of the graph shows that every vertex is joined to three others and so $a_1 = 3$. It follows that $a_2 = 10 - 1 - 3 = 6$.

If $\{\alpha, \beta\}$ is an edge, we readily obtain the partial table

	$C_0(\alpha)$	$C_1(\alpha)$	$C_2(\alpha)$	
$C_0(\beta)$	0	1	0	1
$C_1(\beta)$	1	0		3
$C_2(\beta)$	0			6
	1	3	6	10

because there are no triangles in the graph. To obtain the correct row and column totals, this must be completed as

	$C_0(\alpha)$	$C_1(\alpha)$	$C_2(\alpha)$	
$C_0(\beta)$	0	1	0	1
$C_1(\beta)$	1	0	2	3
$C_2(\beta)$	0	2	4	6
	1	3	6	10

We work similarly for the case that $\{\alpha, \beta\}$ is not an edge. If you are familiar with the Petersen graph you will know that every pair of vertices which are not joined by an edge are both joined to exactly one vertex; if you are not familiar with the graph and its symmetries, you should check that this is true. Thus $p_{11}^2 = 1$. This gives the middle entry of the table, and the three entries in the bottom right-

hand corner (p_{12}^2 , p_{21}^2 and p_{22}^2) can be calculated by subtraction.

	$C_0(\alpha)$	$C_1(\alpha)$	$C_2(\alpha)$	
$C_0(\beta)$	0	0	1	1
$C_1(\beta)$	0	1	2	3
$C_2(\beta)$	1	2	3	6
	1	3	6	10

So here again we have an association scheme with two associate classes. ■

1.2 Graphs

Now that we have done the example with the Petersen graph, we need to examine graphs a little more formally. Recall that a finite graph is a finite set Γ , whose elements are called *vertices*, together with a set of 2-subsets of Γ called *edges*. (A “2-subset” means a subset of size 2.) Strictly speaking, this is an *undirected* graph. Vertices γ and δ are said to be *joined by an edge* if $\{\gamma, \delta\}$ is an edge. The graph is *complete* if every 2-subset is an edge.

Example 1.4 suggests a second way of looking at association schemes. Imagine that all the edges in the Petersen graph are blue. For each pair of distinct vertices which are not joined, draw a red edge between them. We obtain a 2-colouring of the complete undirected graph K_{10} on ten vertices. Condition (iii) gives us a special property about the number of triangles of various types through an edge with a given colour.

Definition (Second definition of association scheme) An *association scheme* with s associate classes on a finite set Ω is a colouring of the edges of the complete undirected graph with vertex-set Ω by s colours such that

(iii)' for all i, j, k in $\{1, \dots, s\}$ there is an integer p_{ij}^k such that, whenever $\{\alpha, \beta\}$ is an edge of colour k then

$$|\{\gamma \in \Omega : \{\alpha, \gamma\} \text{ has colour } i \text{ and } \{\gamma, \beta\} \text{ has colour } j\}| = p_{ij}^k;$$

(iv)' every colour is used at least once;

(v)' there are integers a_i for i in $\{1, \dots, s\}$ such that each vertex is contained in exactly a_i edges of colour i .

The strange numbering is to aid comparison with the previous definition. There is no need for an analogue of condition (i), because every edge consists of two distinct vertices, nor for an analogue of condition (ii), because we have specified that

the graph be undirected. Condition (iii) says that if we fix different vertices α and β , and colours i and j , then the number of triangles which consist of the edge $\{\alpha, \beta\}$ and an i -coloured edge through α and a j -coloured edge through β is exactly p_{ij}^k , where k is the colour of $\{\alpha, \beta\}$, irrespective of the choice of α and β . (See Figure 1.5.) We did not need a condition (iv) in the partition definition because we specified that the subsets in the partition be non-empty. Finally, because condition (iii)' does not deal with the analogue of the diagonal subset, we have to put in condition (v)' explicitly. Alternatively, if we assume that none of the colours used for the edges is white, we can colour all the vertices white. Then we recover exactly the parameters of the first kind, with $a_0 = 1$, $p_{ii}^0 = a_i$, $p_{i0}^i = p_{0i}^i = 1$, and $p_{ij}^0 = p_{i0}^j = p_{0j}^i = 0$ if $i \neq j$.

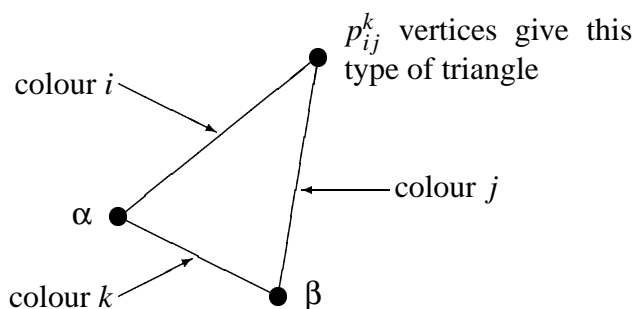


Figure 1.5: Condition (iii)'

Example 1.5 Let Ω be the set of the 8 vertices of the cube. Colour the edges of the cube yellow, the main diagonals red and the face diagonals black. In Figure 1.6 the yellow edges are shown by solid lines, the red edges by dashed lines, and the black edges are omitted. If you find this picture hard to follow, take any convenient cuboid box, draw black lines along the face diagonals and colour the edges yellow.

Every vertex is in three yellow edges, one red edge and three black ones, so $a_{\text{yellow}} = 3$, $a_{\text{red}} = 1$ and $a_{\text{black}} = 3$.

The values p_{ij}^{yellow} are the entries in

		white	yellow	red	black
white	[0	1	0	0
yellow		1	0	0	2
red		0	0	0	1
black		0	2	1	0

],

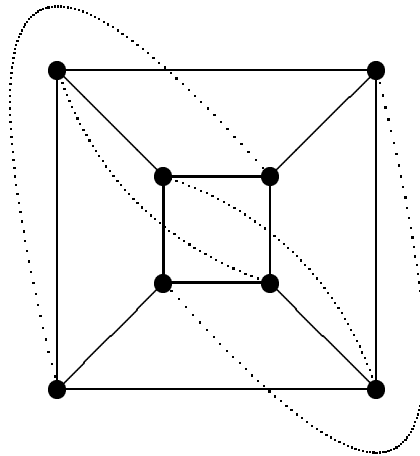


Figure 1.6: The cube: solid lines are yellow, dashed ones are red

the values p_{ij}^{red} are the entries in

	white	yellow	red	black
white	0	0	1	0
yellow	0	0	0	3
red	1	0	0	0
black	0	3	0	0

and

the values p_{ij}^{black} are the entries in

	white	yellow	red	black
white	0	0	0	1
yellow	0	2	1	0
red	0	1	0	0
black	1	0	0	2

.

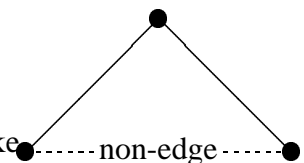
Thus we have an association scheme with three associate classes. ■

If an association scheme has two associate classes, we can regard the two colours as ‘visible’ and ‘invisible’, as in Example 1.4. The graph formed by the visible edges is said to be *strongly regular*.

Definition A finite graph is *strongly regular* if

- (a) it is *regular* in the sense that every vertex is contained in the same number of edges;
- (b) every edge is contained in the same number of triangles;

(c) every non-edge is contained in the same number of configurations like



(d) it is neither complete (all pairs are edges) nor null (no pairs are edges).