QUEEN MARY AND WESTFIELD COLLEGE

MAS 417 — Association Schemes and Partially Balanced Designs

Assignment 1 — **For handing in on 6 February 2001**

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1 Explain why the parameters of an association scheme satisfy

(a) $p^j_{0j} = 1$;

(b) $p^k_{ij} = p^k_{ji}$.

2 Verify that a graph is strongly regular if and only if it is neither complete nor null and the sets of edges and non-edges form an association scheme on the set of vertices. Relate the parameters of the association scheme to those of the strongly regular graph.

3 Draw a finite graph that is regular but not strongly regular.

4 Let A and B be matrices in $F^{\Gamma\times\Delta}$ and $F^{\Delta\times\Phi}$ respectively, where F is a field and Γ, Δ and Φ are finite sets. Prove that $(AB)' = B'A'$.

5 Let $\mathcal{C}_i$ and $\mathcal{C}_j$ be associate classes in an assocation scheme on Ω. Suppose that $(\alpha,\beta) \in \mathcal{C}_i$ and $(\beta,\gamma) \in \mathcal{C}_j$. Prove that there is a point δ such that $(\alpha,\delta) \in \mathcal{C}_j$ and $(\delta,\gamma) \in \mathcal{C}_i$.

6 Consider an association scheme on a set Ω of size 6.

(a) Prove that at most one of the classes can have valency 1.

(b) Write down four distinct association schemes on Ω.

(c) Prove that there are exactly four different association schemes on Ω, in the sense that any others are obtained from one of these four by relabelling.

7 Let A be a symmetric matrix with zero diagonal whose entries are 0 and 1. Suppose that there are integers x, y and z such that $A^2 = xI + yA + zJ$. Show that A, I and $J - A - I$ are the adjacency matrices of an association scheme, and find its parameters.

8 Two Latin squares of the same size are said to be *orthogonal to each other* if each letter of one square occurs exactly once in the same position as each letter of the second square. A collection of Latin squares of the same size is said to be *mutually orthogonal* if every pair of squares in it is orthogonal.

Suppose that $\Lambda_1, \ldots, \Lambda_r$ is a set of r mutually orthogonal Latin squares of size n. Let Ω be the set of n^2 cells in the array. For distinct α, β in Ω, let α and β be first associates if α and β are in the same row or are in the same column or have the same letter in any of $\Lambda_1, \ldots, \Lambda_r$; otherwise α and β are second associates.

(a) Find the size of $\mathcal{C}_1(\alpha)$. Hence find an upper bound on r.

(b) Show that these definitions of $\mathcal{C}_1$ and $\mathcal{C}_2$ make an association scheme on Ω. It is called the *Latin-square type* association scheme $\mathrm{L}(r+2, n)$. When does it have only one associate class?