UNIVERSITY OF LONDON

QUEEN MARY AND WESTFIELD COLLEGE

M. Sc. Examination 1998

A 100 Association Schemes and Partially Balanced Designs

Duration: 3 hours

Date and time:

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Question 1 Define a *distance-regular graph*.

Prove that the distance classes in a distance-regular graph form an association scheme on the set of vertices.

Do all association schemes arise in this way from distance-regular graphs? Justify your answer.

Give two examples of distance-regular graphs with diameter 4.

Question 2 Define a *perfect difference set* for $\mathbf{Z}_n$.

In $\mathbf{Z}_{13}$, let $\Phi = \{0,2,8\}$. Find the concurrence matrix for the cyclic design generated by Φ. Hence find its canonical efficiency factors and the average variance of simple contrasts when the variance of each response is σ^2.

Find a subset Ψ of $\mathbf{Z}_{13}$ such that the cyclic design generated by Φ and Ψ is balanced.

Question 3 Let $\mathcal{Q}_1$ be an association scheme on a set Ω_1 of size n_1 with s_1 non-diagonal associate classes $\mathcal{C}_1, \ldots, \mathcal{C}_{s_1}$, and let $\mathcal{Q}_2$ be an association scheme on a set Ω_2 of size n_2 with s_2 non-diagonal associate classes $\mathcal{D}_1, \ldots, \mathcal{D}_{s_2}$. Define the *wreath product* $\mathcal{Q}_1/\mathcal{Q}_2$.

Prove that $\mathcal{Q}_1/\mathcal{Q}_2$ is an association scheme, and find its characters in terms of those of $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

Describe the association scheme $\underline{5}/(\underline{2} \times \underline{4})$ in words or pictures.

Question 4 Describe the triangular association scheme $\mathrm{T}(n)$ and prove that it *is* an association scheme.

Find the stratum projectors of $\mathrm{T}(6)$ in terms of the adjacency matrices.

Question 5 For each of the sets of values of n, a, p and q below, either describe a strongly regular graph on n vertices with valency a such that every edge is contained in exactly p triangles and every non-edge is contained in exactly q paths of length 2, or prove that no such strongly regular graph exists.

(a) $n = 9$, $a = 4$, $p = 1$, $q = 2$.

(b) $n = 20$, $a = 8$, $p = 1$, $q = 4$.

(c) $n = 25$, $a = 6$, $p = 2$, $q = 1$.

(d) $n = 36$, $a = 20$, $p = 10$, $q = 12$.

Question 6 Given an incomplete-block design for treatment set Θ, define what it means for the design to be *partially balanced*.

Prove that, for a partially balanced incomplete-block design, the variance of the simple contrast for the difference between treatments θ and η is a function of the associate class containing (θ, η).

Show that the following design for $\Theta = \{a,b,c,d,e,f,g,h,i,j,k,l\}$ is partially balanced, and find the variances of simple contrasts when the variance of each response is σ^2.

$\{a,e,i\}$, $\{b,f,j\}$, $\{c,g,k\}$, $\{d,h,l\}$, $\{a,h,k\}$, $\{b,g,l\}$, $\{c,f,i\}$, $\{d,e,j\}$,
$\{a,g,j\}$, $\{b,h,i\}$, $\{c,e,l\}$, $\{d,f,k\}$, $\{a,f,l\}$, $\{b,e,k\}$, $\{c,h,j\}$,$\{d,g,i\}$.

Question 7 Define a *difference partition* for $\mathbf{Z}_n$.

Show that $\{0\}$, $\{5,-5\}$, $\{1,4,6,-6,-4,-1\}$ and $\{2,3,7,-7,-3,-2\}$ form a difference partition for $\mathbf{Z}_{15}$.

Find the character table of the association scheme defined by this difference partition.

 END OF EXAMINATION