QUEEN MARY, UNIVERSITY OF LONDON

MAS 305 — Algebraic Structures II

Assignment 4 — For handing in on 1 November 2006

Write your name and student number at the top of your assignment before handing it in. Staple all the pages together. Give the completed assignment to me in person in the Wednesday lecture.

1 Let g and h be elements of a group G, with orders n and m respectively. Prove that if $gh = hg$ and $\langle g \rangle \cap \langle h \rangle = \{1_G\}$ then the order of gh is $\mathrm{lcm}(n,m)$.

2 In a group G, let g be an element of order pq, where p and q are distinct primes. Prove that G has subgroups H and K of orders p and q respectively such that $\langle g \rangle = H \times K$.

3 Let K be a subgroup of a group G such that $K \subseteq Z(G)$.

(a) Prove that $K \trianglelefteq G$.

(b) Prove that if G/K is cyclic then G is Abelian.

4 Let H be a subgroup of a group G, and let g be an element of G. Put $H^g = \{g^{-1}hg : h \in H\}$.

(a) Prove that H^g is a subgroup of G.

(b) Define $\theta \colon H \to H^g$ by $h\theta = g^{-1}hg$ for h in H. Prove that θ is an isomorphism.

5 Let Ω be the set of all subgroups of a group G. For g in G, define π_g by $H\pi_g = H^g$ for H in Ω. Prove that π is an action of G.

6 Let Ω be the set of integers modulo 5. Let G be the set of all permutations of Ω of the form

$$x \mapsto ax + b,$$

where a and b are in $\mathbb{Z}_5$ and $a \neq 0$. You may assume that G is a subgroup of S_5.

(a) Find G_0, the stabilizer of 0 in G.

(b) Hence or otherwise, find out how many elements G has of each cycle structure.

(c) Let $H = \langle h \rangle$, where $h \colon x \mapsto x + 1$. Prove that $N(G) \subseteq N(H)$, where both normalizers are taken in S_5.