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UNIVERSITY OF LONDON

B. Sc. Examination 2007

MAS 305 Algebraic Structures II

Duration: 2 hours

Date and time: 22 May 2007, 1000h

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Do not start reading the question paper until instructed to do so.

The question paper must not be removed from the examination room.

 TURN OVER

Question 1 (a) Let G be a finite group, and let p be a prime number. State what it means for G to be a *p-group*. Define a *Sylow p-subgroup* of G. State what it means for G to be *simple*.

(b) State all the parts of Sylow's Theorems (it does not matter how the parts are numbered).

(c) Prove that no group of order 80 is simple.

(d) Describe the Sylow 2-subgroups of the dihedral group D_{24} of all symmetries of the regular polygon with 12 sides.

Question 2 (a) State what it means for a finite group G to be *soluble*.

(b) Show that the symmetric group S_4 is soluble.

(c) Prove that if H is a subgroup of S_4 and $|H| = 12$ then H is the alternating group A_4.

(d) Let G be a group of order 12, and P a Sylow 3-subgroup of G. Prove that if P is not normal in G then $G \cong A_4$.

Question 3 (a) Let π be an action of a group G on a set Ω, and let α and β be elements of Ω. Define the *stabilizer* G_α of α in G. Prove that if $\alpha\pi_g = \beta$ for some g in G then $G_\alpha^g = G_\beta$.

(b) Let Ω be $\mathbb{Z}_7$, the set of integers modulo 7. Let G consist of all permutations of Ω of the form

$$x \mapsto ax + c \quad \text{modulo } 7,$$

where a is equal to 1, 2 or 4 in $\mathbb{Z}_7$ and $c \in \mathbb{Z}_7$. You may assume that G is a subgroup of S_7.

(i) Find the stabilizer G_0 of 0 in G.

(ii) Hence or otherwise, find the conjugacy classes of G: this means giving one element of each conjugacy class explicitly and finding out how many elements are conjugate to it. For one element g in each conjugacy class, find its centralizer $C(g)$ in G.

Question 4 Let $(G, +)$ be an Abelian group with (additive) identity 0_G. Define the binary operation $*$ on G by $g * h = 0_G$ for all g, h in G.

(a) Prove that $(G, +, *)$ is a commutative ring. From now on, we call this ring R.

(b) Does R have an identity?

(c) Describe the ideals of R.

(d) For each of the following rings, state whether or not it is a zero ring, and justify your answer: (i) $M_2(R)$, (ii) $R \oplus R$, (iii) $R[x]$.

(e) Identify the additive group of (i) $M_2(R)$, (ii) $R \oplus R$.

(f) Prove that if $|R| = 64$ then R has an ideal of order 16.

Question 5 Let R be a principal ideal domain.

(a) Let u, x, y and h be elements of R. Explain what each of the following statements means:

(i) u is a *unit* in R;

(ii) h is a *highest common factor* of x and y.

(b) Let A be a 2×2 matrix with entries in R. Prove that A is invertible if and only if $\det(A)$ is a unit in R.

(c) Let G be the group of all 2×2 invertible matrices with entries in R, and let

$$\Omega = \{(x, y) : x \in R,\ y \in R\} = R^2.$$

For g in G, define $\pi_g\colon \Omega \to \Omega$ by $\alpha\pi_g = \alpha g$ for α in Ω; that is, if $\alpha = (x, y)$ and $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\alpha\pi_g = (x, y) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (ax + cy, bx + dy)$.

(i) Show that π is an action of G on Ω.

(ii) Show that (x, y) is in the same orbit as $(1_R, 0_R)$ if and only if 1_R is a highest common factor of x and y.

Question 6 Let R be a ring with identity 1.

(a) What is meant by an *ascending chain of ideals* of R? What does it mean to say that R is *Noetherian*?

(b) In the matrix ring $M_2(R)$, let E_{ij} be the matrix whose (i, j)-entry is 1, all other entries being 0. Let A be the matrix whose (i, j)-entry is a_{ij}. Here $1 \leq i, j \leq 2$. Prove that $E_{ij}AE_{kl} = a_{jk}E_{il}$.

(c) Let J be a non-zero ideal of $M_2(R)$. Prove that there is some ideal I of R such that $J = M_2(I)$.

(d) Hence or otherwise, prove that if R is Noetherian then $M_2(R)$ is Noetherian.

(e) How does the proof that $M_2(R)$ is Noetherian break down if R does not contain an identity?

Question 7 (a) Let A and I be subsets of a ring R. What do each of the following statements mean?

(i) I is an *ideal* of R;

(ii) I is the ideal $\langle A \rangle$ *generated* by A?

(iii) I is a *finitely generated* ideal.

(b) Let R be a ring.

(i) What is meant by a *maximal* element of a non-empty set of ideals of R?

(ii) Prove that if every non-empty set of ideals of R has a maximal element then every ideal of R is finitely generated.

(c) Prove that every non-empty set of ideals of the ring $\mathbb{Z}[x]$ has a maximal element, citing any theorems that you use.

 END OF EXAMINATION