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UNIVERSITY OF LONDON

B. Sc. Examination 2006

MAS 305 Algebraic Structures II

Duration: 2 hours

Date and time: 4 May 2006, 1430h

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

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Question 1 (a) State what it means for a finite group G to be *soluble*.

(b) Let G be a finite group. Prove that if G has a normal subgroup N such that N and G/N are both soluble then G is soluble.

(c) Prove that every group of order 48 is soluble.

Question 2 (a) Define what is meant by an *action* of a group G on a set Ω; by an *orbit* of such an action; and by a *fixed point* of such an action.

(b) Let H be a subgroup of a group G. Define the *normalizer* $N(H)$ of H in G.

(c) Let H be a subgroup of a group G. Let Ω be the set of right cosets of H in G. For h in H, define π_h by

$$\pi_h \colon Hg \mapsto Hgh$$

for Hg in Ω. Prove that

(i) π is an action of H on Ω;

(ii) the coset Hg is a fixed point of this action if and only if $g \in N(H)$;

(iii) if G is finite, $|H|$ is a power of a prime p and p divides $|G : H|$ then $N(H)$ is strictly bigger than H.

Question 3 (a) Let x be an element of a group G. Define what is meant by (i) the *conjugacy class* of x in G, and (ii) the *centralizer* of x in G. State a theorem which connects these two concepts when G is finite.

(b) Let G be the group D_{24} of all symmetries of the regular polygon with 12 sides. Describe the elements of G. Find all the conjugacy classes. Also find the centralizer of each element.

(c) How many Sylow 2-subgroups does D_{24} have? Justify your answer.

Question 4 For each of the following statements, say whether it is true or false. If it is true, prove it. If it is false, give a counter-example, explaining *why* it is a counterexample.

(a) If G is an Abelian group and N is a normal subgroup of G then G/N is Abelian.

(b) If G and H are cyclic groups then $G \times H$ is a cyclic group.

(c) If S is a subring of a ring R and S has an identity then R has an identity.

(d) If R is a principal ideal domain and a and b are elements of R then a and b have a highest common factor.

Question 5 (a) What does it mean to say that (i) a ring R is *simple*, (ii) a group G is *simple*?

(b) Let F be a field. Explain what the sets $M_2(F)$ and $\mathrm{GL}(2, F)$ are. Under what operations is $M_2(F)$ a ring? Under what operation is $\mathrm{GL}(2, F)$ a group?

(c) Prove that if F is a field then $M_2(F)$ is a simple ring.

(d) Show that if F is a field then $\mathrm{GL}(2, F)$ is not a simple group.

Question 6 Let R be a ring, let I be an ideal of R and let S be a subring of R. Let $I + S = \{x + y : x \in I,\ y \in S\}$. Prove that (a) $I \cap S$ is an ideal of S; (b) $I + S$ is a subring of R, and (c) $S/I \cap S$ is isomorphic to another quotient ring defined in terms of I and S, which you should identify.

Apply the last result in the case that $R = \mathbb{Z}$, $I = 4\mathbb{Z}$ and $S = 10\mathbb{Z}$.

Question 7 (a) Let R be a ring. What is meant by an *ascending chain of ideals* of R? What does it mean to say that R is *Noetherian*? What is meant by a *maximal element* of a non-empty set of ideals of R?

(b) Prove that if the ring R is Noetherian and I is an ideal of R then R/I is Noetherian.

(c) Prove that if the ring R is Noetherian then every non-empty set of ideals of R contains a maximal element.

(d) State Hilbert's Basis Theorem.

(e) Let θ be the real cube root of 2 and put $S = \left\{a + b\theta + c\theta^2 : a, b, c \in \mathbb{Z}\right\}$. You may assume that S is a ring. Starting from Hilbert's Basis Theorem, show that S is Noetherian.

 END OF EXAMINATION