MAS 305 **Algebraic Structures II**

Notes 6 **Autumn 2006**

Groups of prime-power order

Definition Let p be a prime. A finite group G is a *p-group* if $|G|$ is a power of p.

For example, D_8 is a 2-group.

Lagrange's Theorem shows that if G is a p-group and g is an element of G then the order of g is a power of p.

Theorem If G is a non-trivial finite p-group for some prime p then $Z(G) \neq \{1_G\}$.

Proof Let $|G| = p^n$ for some $n \geqslant 1$. Every conjugacy class in G has size dividing p^n, so has size p^r for some $r \leqslant n$. Suppose that there are m_r conjugacy classes of size p^r for $r = 0, 1, \ldots, n$. Then

$$m_0 1 + m_1 p + m_2 p^2 + \cdots + m_r p^r + \cdots + m_n p^n = p^n,$$

so p divides m_0. But $\{x\}$ is a whole conjugacy class of size 1 if and only if $x \in Z(G)$, so $m_0 = |Z(G)|$: therefore $m_0 \neq 0$, because $\{1_G\} \leqslant Z(G)$. So $|Z(G)|$ is a non-zero multiple of p, and therefore $|Z(G)| \geqslant p$. $\square$

(Compare this with the proof of Cauchy's Theorem.)

Corollary If G is a finite group of order p^n, where p is prime, then there are subgroups

$$\{1_G\} = G_0 < G_1 < \cdots < G_n = G$$

such that $|G_i| = p^i$ and $G_i \trianglelefteq G$ for $i = 0, \ldots, n$.

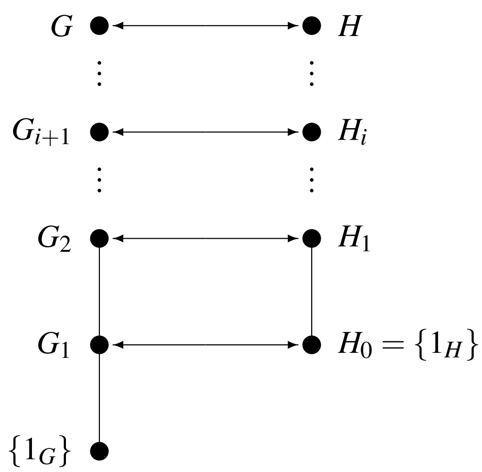
Proof The proof is by induction on n. The statement is true when $n = 1$, for then $G_0 = \{1_G\}$ and $G_1 = G$.

Now take $n \geqslant 2$, and assume that the statement is true for $n-1$. The theorem says that $Z(G) \neq \{1_G\}$, so p divides $|Z(G)|$. By Cauchy's Theorem, $Z(G)$ has an element z of order p. Put $G_1 = \langle z \rangle$. Then $G_1 \trianglelefteq G$, because $G_1 \leqslant Z(G)$. Also, $|G_1| = p$.

Put $H = G/G_1$. Then $|H| = p^n/p = p^{n-1}$, so by the inductive hypothesis H has subgroups

$$\{1_H\} = H_0 < H_1 < \cdots < H_{n-1} = H$$

with $|H_i| = p^i$ and $H_i \trianglelefteq H$ for $i = 0, \ldots, n-1$. By the Correspondence Theorem, there is a subgroup G_{i+1} of G containing G_1 such that $G_{i+1}/G_1 = H_i$ and $G_{i+1} \trianglelefteq G$ for $i = 0, \ldots, n-1$. Moreover, $|G_{i+1}| = |G_1| \times |H_i| = p^{i+1}$ for $i = 0, \ldots, n-1$. Finally, the Correspondence Theorem shows that $G_i \leqslant G_{i+1}$ for $i = 0, \ldots, n-1$. $\square$



Theorem If G is Abelian then $Z(G) = G$; otherwise, $G/Z(G)$ is not cyclic.

Proof Part of the coursework.

Small p-groups

Let p be a prime. If $|G| = p$ then G is cyclic, because Lagrange's Theorem shows that every element of G other than the identity has order p.

If $|G| = p^2$ then $|Z(G)|$ is 1 or p or p^2. We have just proved that $|Z(G)| \neq 1$. If $|Z(G)| = p$ then $|G/Z(G)| = p$ so $G/Z(G)$ is cyclic, contradicting the above theorem: hence $|Z(G)| \neq p$. Therefore $Z(G) = G$ and so G is Abelian.

If G has an element of order p^2 then it is cyclic. Otherwise, all non-identity elements have order p. Let a be an element of order p, and put $A = \langle a \rangle$. Choose any element b in $G \setminus A$. Then b also has order p. Put $B = \langle b \rangle$. Now $A \cap B \leqslant B$; and $A \cap B$ cannot be B, because $b \notin A$, so $A \cap B = \{1_G\}$. Moreover, $xy = yx$ for all x in A and all y in B, because G is Abelian. Therefore G contains the internal direct product $\langle a \rangle \times \langle b \rangle$. Because of the uniqueness of the expression of an element of an internal direct product,

$$\langle a \rangle \times \langle b \rangle = \{a^n b^m : 0 \leqslant n \leqslant p-1,\ 0 \leqslant m \leqslant p-1\},$$

and these p^2 products are all distinct. Therefore $G = \langle a \rangle \times \langle b \rangle$.

Challenge!

Find all groups of order 8.

Infinite p-groups

What could an infinite p-group be? Here is an example of an infinite group in which every element has order a power of the prime 2. We work inside the infinite Abelian group $(\mathbb{C} \setminus \{0\}, \times)$. Put

$$G = \left\{ \mathrm{e}^{2\pi \mathrm{i} m/2^n} : n \in \mathbb{Z},\ n \geqslant 0,\ m \in \mathbb{Z} \right\}.$$

Then G contains elements of order 2^n for alll non-negative integers n.