MAS 305 Algebraic Structures II

Notes 11 Autumn 2006

Ring homomorphisms

Definition Let R and S be rings, and let $\phi\colon R \to S$ be a function. Then ϕ is a *ring homomorphism* if

$$(r_1 + r_2)\phi = r_1\phi + r_2\phi$$

and

$$(r_1 r_2)\phi = (r_1\phi)(r_2\phi)$$

for all r_1, r_2 in R.

A ring homomorphism which is a bijection is called an *isomorphism*. If there is an isomorphism ϕ from R to S then ϕ^{-1} is also an isomorphism and R is *isomorphic* to S, written $R \cong S$.

Definition If $\phi\colon R \to S$ is a ring homomorphism then the *image* of ϕ is $\{r\phi : r \in R\}$, written $\mathrm{Im}(\phi)$, and the *kernel* of ϕ is $\{r \in R : r\phi = 0_S\}$, written $\ker(\phi)$.

Theorem If $\phi\colon R \to S$ is a ring homomorphism then

(a) $\mathrm{Im}(\phi)$ is a subring of S;

(b) $\ker(\phi)$ is an ideal of R;

(c) $r_1\phi = r_2\phi$ if and only if r_1 and r_2 are in the same coset of $\ker(\phi)$.

Proof (a) We know that $(\mathrm{Im}(\phi), +)$ is a subgroup of $(S, +)$, from the similar theorem for groups. If s_1 and s_2 are in $\mathrm{Im}(\phi)$ then there are r_1, r_2 in R with $r_1\phi = s_1$ and $r_2\phi = s_2$. Then $s_1 s_2 = (r_1\phi)(r_2\phi) = (r_1 r_2)\phi \in \mathrm{Im}(\phi)$, so $\mathrm{Im}(\phi) \leqslant S$.

(b) We know that $(\ker(\phi), +)$ is a subgroup of $(R, +)$, from the group theory. If $r \in \ker(\phi)$ and $t \in R$ then $(rt)\phi = (r\phi)(t\phi) = 0_S(t\phi) = 0_S$ and $(tr)\phi = (t\phi)(r\phi) = (t\phi)0_S = 0_S$. Thus $rt \in \ker(\phi)$ and $tr \in \ker(\phi)$. Therefore $\ker(\phi) \trianglelefteq R$.

(c) We know this because $\phi\colon (R,+) \to (S,+)$ is a group homomorphism. $\square$

The theorem has been stated in this way because parts (a) and (b) are so important. However, essentially the same proof can generalize part (a) to the image of any subring of R, and generalize part (b) to the inverse image of any subring or ideal of $\mathrm{Im}(\phi)$. We shall do this in the Correspondence Theorem.

Theorem Let I be an ideal of a ring R. The function $\theta\colon R \to R/I$ defined by $r\theta = I + r$ is a ring homomorphism (called the *canonical* homomorphism for I) and its kernel is I.

Proof For all r, s in R:

$$r\theta + s\theta = (I+r) + (I+s) = I + (r+s) = (r+s)\theta$$

and

$$(r\theta)(s\theta) = (I+r)(I+s) = I + rs = (rs)\theta,$$

so θ is a ring homomorphism. Moreover,

$$r \in \ker(\theta) \iff r\theta = I \iff I + r = I \iff r \in I. \quad \square$$

First Isomorphism Theorem for Rings If R and S are rings and $\phi\colon R \to S$ is a ring homomorphism then $R/\ker(\phi) \cong \mathrm{Im}(\phi)$.

Proof Put $I = \ker(\phi)$. Define $\psi\colon R \to \mathrm{Im}(\phi)$ by $(I+r)\psi = r\phi$ for r in R. We know that $r_1\phi = r_2\phi$ if and only if $I + r_1 = I + r_2$, so ψ is well defined and one-to-one. Clearly ψ is onto.

Furthermore,

$$\begin{aligned} (I+r_1)\psi + (I+r_2)\psi &= r_1\phi + r_2\phi \\ &= (r_1+r_2)\phi \\ &= [I + (r_1+r_2)]\psi \\ &= [(I+r_1) + (I+r_2)]\psi \end{aligned}$$

and

$$\begin{aligned} [(I+r_1)\psi][(I+r_2)\psi] &= (r_1\phi)(r_2\phi) \\ &= (r_1r_2)\phi \\ &= (I + r_1r_2)\psi \\ &= [(I+r_1)(I+r_2)]\psi \end{aligned}$$

for all r_1, r_2 in R, so ψ is a homomorphism. $\square$

Some authors include parts (a) and (b) of the first theorem in this section in the statement of the First Isomorphism Theorem.

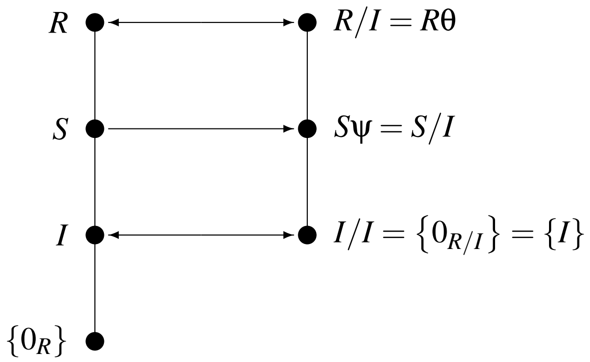
Lemma if $\phi_1 : R_1 \to R_2$ and $\phi_2 : R_2 \to R_3$ are ring homomorphisms then $\phi_1\phi_2 : R_1 \to R_3$ is a ring homomorphism.

Proof Exercise.

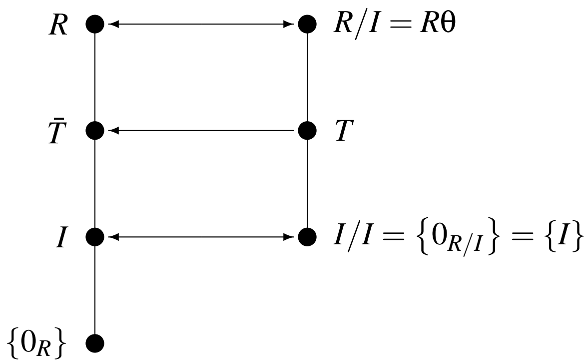
Correspondence Theorem for Rings Let I be an ideal of a ring R. There is a bijection ψ from the set of subrings of R containing I and the set of subrings of R/I. The bijection preserves inclusion ($S_1 \subseteq S_2 \iff S_1\psi \subseteq S_2\psi$), and ideals of R containing I correspond to ideals of R/I.

Proof Define $S\psi = \{I+s : s \in S\} = S/I$, for subrings S of R containing I. Let θ be the canonical homomorphism for I. Given S, let $\tilde{\theta}$ be the restriction of θ to S. Then $\tilde{\theta}$ is a homomorphism, and $\mathrm{Im}(\tilde{\theta}) = \{s\theta : s \in S\} = \{I+s : s \in S\} = S\psi$. But $\mathrm{Im}(\tilde{\theta})$ is a subring of R/I, so $S\psi$ is a subring of R/I.

Given $S\psi$, we can recover S as the union of the cosets of I which are elements of $S\psi$, so ψ is one-to-one. Clearly, ψ preserves inclusion.



Let T be a subring of R/I. Put $\bar{T} = \{r \in R : I+r \in T\}$. We know from the group theory that $(\bar{T},+)$ is a subring of $(R,+)$ containing I. If r_1, r_2 are in $\bar{T}$ then $I+r_1 \in T$ and $I+r_2 \in T$, so $(I+r_1)(I+r_2) \in T$, so $I+r_1r_2 \in T$, so $r_1r_2 \in \bar{T}$. Hence $\bar{T}$ is a subring of R. Clearly, $\bar{T}\psi = T$. Therefore ψ is onto.



Finally,

$$\begin{aligned} S\psi \trianglelefteq R/I &\iff (I+s)(I+r) \in S\psi \quad \text{for all } s \text{ in } S \text{ and all } r \text{ in } R \\ &\quad\text{and}\quad (I+r)(I+s) \in S\psi \quad \text{for all } s \text{ in } S \text{ and all } r \text{ in } R \\ &\iff I+sr \in S\psi \text{ and } I+rs \in S\psi \quad \text{for all } s \text{ in } S \text{ and all } r \text{ in } R \\ &\iff sr \in S \text{ and } rs \in S \quad \text{for all } s \text{ in } S \text{ and all } r \text{ in } R \\ &\iff S \trianglelefteq R. \quad \square \end{aligned}$$

Second Isomorphism Theorem for Rings If I and J are ideals of a ring R with $I \leqslant J$ then $(R/I)/(J/I) \cong R/J$.

Proof Exactly like the proof of the Second Isomorphism Theorem for groups. □

Some authors include the Corrspondence Theorem in the statement of the Second Isomorphism Theorem.

Third Isomorphism Theorem for Rings If R is a ring, I is an ideal of R and S is a subring of R, define $I+S = \{x+y : x \in I,\ y \in S\}$. Then

(a) $I+S$ is a subring of R containing I;

(b) $I \cap S$ is an ideal of S;

(c) $(I+S)/I \cong S/I \cap S$.

Proof A small adaptation of the proof of the Third Isomorphism Theorem for groups. □