MAS 305 Algebraic Structures II

Notes 1 **Autumn 2006**

Review of Equivalence Relations

A binary relation $\sim$ on a set A is an *equivalence relation* if

(a) $\sim$ is *reflexive*, which means that $a \sim a$ for all a in A;

(b) $\sim$ is *symmetric*, which means that if $a \sim b$ then $b \sim a$;

(c) $\sim$ is *transitive*, which means that if $a \sim b$ and $b \sim c$ then $a \sim c$.

Example Take A to be the set $\mathbb{Z}$ of integers. Given a fixed integer m, define $a \sim_m b$ if m divides $a - b$.

(a) m divides 0, so $a \sim_m a$ for all a, so $\sim_m$ is reflexive;

(b) if m divides $a - b$ then m divides $b - a$, so $\sim_m$ is symmetric;

(c) if m divides $a - b$ and m divides $b - c$ then m divides $(a - b) + (b - c)$, which is $a - c$, so $\sim_m$ is transitive.

Given an equivalence relation $\sim$, the *equivalence class* containing a is

$$\{b \in A : b \sim a\}.$$

Theorem The equivalence classes form a *partition* of A, in the sense that each element of A belongs to exactly one equivalence class.

Thus two equivalence classes are either exactly the same or disjoint.

Notation In general, write $[a]$ for the equivalence class containing a.

Example In $\mathbb{Z}$, the equivalence classes of $\sim_4$ are:

$$\begin{aligned}
\{\ldots,-4,0,4,8,\ldots\} &= [0]\\
\{\ldots,-3,1,5,9,\ldots\} &= [1]\\
\{\ldots,-6,-2,2,6,10,\ldots\} &= [2]=[10]\\
\{\ldots,-5,-1,3,7,\ldots\} &= [3]=[-1].
\end{aligned}$$

Notation This set of four classes is called $\mathbb{Z}/(4)$ or $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}_4$.

When we manipulate equivalence classes, we have to make sure that our definitions do not depend on the names we have given to them. For example, in addition in $\mathbb{Z}_4$, it makes no difference whether we refer to $[3]$ or to $[-1]$.