Chapter 3

Simple treatment structure

3.1 Replication of control treatments

Suppose that treatment 1 is a control treatment and that treatments 2, ..., t are new treatments which we want to compare with the control. Then we want to estimate $\tau_i - \tau_1$ for $i = 2, \ldots, t$. From Equation (2.2), the average variance of these estimators is equal to

$$\frac{1}{t-1}\sum_{i=2}^{t}\left(\frac{1}{r_1}+\frac{1}{r_i}\right) = \frac{1}{r_1}+\frac{1}{t-1}\sum_{i=2}^{t}\frac{1}{r_i}.$$

For given values of r_1 and N, Proposition 2.7 shows that this average variance is minimized when $r_2 = r_3 = \cdots = r_t = (N-r_1)/(t-1)$.

Put $r_1 = r$ and

$$g(r) = \frac{1}{r}+\frac{1}{t-1}\sum_{i=2}^{t}\frac{t-1}{N-r} = \frac{1}{r}+\frac{t-1}{N-r},$$

so that we need to choose r to minimize $g(r)$. Now, g is differentiable on $(0,N)$ and increases without limit as $r \to 0$ and as $r \to N$. Moreover,

$$g'(r) = -\frac{1}{r^2}+\frac{(t-1)}{(N-r)^2},$$

which is zero when, and only when, $r = (N-r)/\sqrt{t-1}$. Thus $g(r)$ is minimized when, and only when, $r = (N-r)/\sqrt{t-1}$; that is

$$r_1 = r = (t-1)r_2/\sqrt{t-1} = \sqrt{t-1}\,r_2. \tag{3.1}$$

In practice we have to use approximate solutions to Equation (3.1) because all the replications must be integers.

Sometimes there is more than one control treatment, and we want to compare every new treatment with every control treatment. Proposition 2.7 shows that all the control treatments should have the same replication as each other, say r_1, while all the new treatments should have the same replication as each other, say r_t. If there are n control treatments and m new treatments we then have $nr_1 + mr_t = N$, and need to minimize

$$\frac{1}{r_1} + \frac{1}{r_t}$$

subject to this constraint. Put

$$g(r) = \frac{1}{r} + \frac{m}{N - nr}.$$

Then

$$g'(r) = -\frac{1}{r^2} + \frac{nm}{(N - nr)^2},$$

which is zero when $r = (N - nr)/\sqrt{nm}$. Thus the average variance of estimators of differences between control treatments and new treatments is minimized when

$$\sqrt{n}\, r_1 = \sqrt{n}\, r = (N - nr)/\sqrt{m} = \sqrt{m}\, r_t. \tag{3.2}$$

Example 3.1 (Example 1.15 continued: Oilseed rape) In this experiment there were two control treatments and eight new treatments. Equation (3.2) gives $\sqrt{2}\, r_1 = \sqrt{8}\, r_t$, so the replication of the controls should have been twice that of the new treatments. In fact, all treatment were applied to two plots each. Perhaps the comparisons between pairs of new treatments were deemed as interesting as those between new treatments and controls?

There are some experiments with a single control treatment where it is known in advance that the control treatment is ineffective. In such circumstances, comparisons between new treatments are more informative then any comparison between a new treatment and the control. This suggests that the new treatments should have higher replication than the control. However, the person who wants to include the control treatment probably wants to compare all new treatments with it, rather than among themselves, so will want higher replication for the control. You may have to compromise on equal replication.

3.2 Comparing new treatments in the presence of a control

Suppose that treatment 1 is a control treatment and that treatments 2, ..., t are new treatments. Rather than asking

is $\tau_i = \tau_1$?

for $i = 2, \ldots, t$, we could ask the two questions

(i) is $\tau_2 = \tau_3 = \cdots = \tau_t$?

(ii) is τ_1 equal to the average of $\tau_2, \ldots, \tau_t$?

To test the null hypothesis

$$\mathrm{H}_0 : \tau_2 = \tau_3 = \cdots = \tau_t$$

against the alternative hypothesis

$$\mathrm{H}_1 : \tau_2, \ldots, \tau_t \text{ are not all equal,}$$

we calculate the mean square for the non-control treatments, which is equal to $\mathrm{SS}(\text{new})/(t-2)$ where

$$\mathrm{SS}(\text{new}) = \sum_{i=2}^{t} \frac{\mathsf{sum}_{T=i}^2}{r_i} - \frac{\left(\sum_{i=2}^{t} \mathsf{sum}_{T=i}\right)^2}{N - r_1}. \tag{3.3}$$

This is then compared with the residual mean square. If we decide that there is a constant ϕ such that $\tau_i = \phi$ for $i = 2, \ldots, t$ then it is reasonable to go on and test the null hypothesis

$$\mathrm{H}_0' : \tau_1 = \phi$$

against the alternative hypothesis

$$\mathrm{H}_1' : \tau_1 \neq \phi.$$

We obtain the sum of squares for this by pretending that all the new treatments are a single treatment: it is equal to

$$\frac{\mathsf{sum}_{T=1}^2}{r_1} + \frac{\left(\sum_{i=2}^{t} \mathsf{sum}_{T=i}\right)^2}{N - r_1} - \frac{\mathsf{sum}^2}{N}. \tag{3.4}$$

These tests can be related to a chain of vector subspaces similar to the one in Figure 2.3. Let C be the "control" factor on Ω: it is defined by

$$C(\omega) = \begin{cases} 1 & \text{if } T(\omega) = 1 \\ 2 & \text{otherwise.} \end{cases}$$

Define the subspace V_C of V to consist of all those vectors $\mathbf{v}$ for which $v_\alpha = v_\beta$ whenever $C(\alpha) = C(\beta)$. Then $\dim V_C = 2$ and $V_0 \subset V_C \subset V_T$. Further, define

$$W_C = V_C \cap V_0^\perp,$$

which has dimension 1, and

$$W_T' = V_T \cap V_C^\perp,$$

$$
\begin{array}{lll}
\mathbb{E}(Y_\omega) = \tau_{T(\omega)} & \mathbb{E}(\mathbf{Y}) \in V_T & V_T \\
 & & \Big| \qquad \Big\} W_T' \\
\mathbb{E}(Y_\omega) = \begin{cases} \tau_1 & \text{if } T(\omega) = 1 \\ \phi & \text{otherwise} \end{cases} & \mathbb{E}(\mathbf{Y}) \in V_C & V_C \\
 & & \Big| \qquad \Big\} W_C \\
\mathbb{E}(Y_\omega) = \kappa & \mathbb{E}(\mathbf{Y}) \in V_0 & V_0 \\
 & & \Big| \qquad \Big\} V_0 \\
\mathbb{E}(Y_\omega) = 0 & \mathbb{E}(\mathbf{Y}) \in \{\mathbf{0}\} & \{\mathbf{0}\}
\end{array}
$$

Figure 3.1: Four models for the expectation of $\mathbf{Y}$.

which has dimension $t-2$. Then the null hypothesis H_0 corresponds to the model

$$\mathbb{E}(\mathbf{Y}) \in V_C,$$

while H_0' corresponds to

$$P_{W_C}(\mathbb{E}(\mathbf{Y})) = \mathbf{0}.$$

Thus we obtain the chain of models shown in Figure 3.1.

To test H_0, we look at the size of $P_{V_T}\mathbf{y} - P_{V_C}\mathbf{y}$, which is equal to $P_{W_T'}\mathbf{y}$. Since

$$P_{V_C}\mathbf{y} = \mathsf{mean}_{C=1}\mathbf{u}_1 + \mathsf{mean}_{C=2}(\mathbf{u}_0 - \mathbf{u}_1),$$

we have

$$
\begin{aligned}
\mathrm{SS}(W_T') = \left\| P_{W_T'}\mathbf{y} \right\|^2 &= \|P_{V_T}\mathbf{y}\|^2 - \|P_{V_C}\mathbf{y}\|^2 \\
&= \sum_{i=1}^{t} \frac{\mathsf{sum}_{T=i}^2}{r_i} - \frac{\mathsf{sum}_{C=1}^2}{r_1} - \frac{\mathsf{sum}_{C=2}^2}{N-r_1} \\
&= \sum_{i=2}^{t} \frac{\mathsf{sum}_{T=i}^2}{r_i} - \frac{\mathsf{sum}_{C=2}^2}{N-r_1},
\end{aligned}
$$

as in Equation (3.3). In words, the

$$
\begin{aligned}
\text{sum of squares for new treatments} \;=\; & \text{crude sum of squares for treatments} \\
& - \text{crude sum of squares for control.}
\end{aligned}
$$

Similarly, to test H_0', we look at the size of $P_{W_C}\mathbf{y}$, which is equal to $P_{V_C}\mathbf{y} - P_{V_0}\mathbf{y}$. We have

$$
\begin{aligned}
\mathrm{SS}(W_C) = \|P_{W_C}\mathbf{y}\|^2 &= \|P_{V_C}\mathbf{y}\|^2 - \|P_{V_0}\mathbf{y}\|^2 \\
&= \frac{\mathsf{sum}_{C=1}^2}{r_1} + \frac{\mathsf{sum}_{C=2}^2}{N-r_1} - \frac{\mathsf{sum}^2}{N},
\end{aligned}
$$

source	sum of squares	degrees of freedom
mean	CSS(mean)	1
control	CSS(control) − CSS(mean)	1
new treatments	CSS(treatments) − CSS(control)	$t-2$
residual	← ..... by subtraction ..... →	
Total	$\sum_\omega y_\omega^2$	N

Table 3.1: First three column of anova table when there is one control treatment and no other structure

as in Equation (3.4). In words, the

$$\begin{aligned}\text{sum of squares for control} &= \text{crude sum of squares for control} \\ &\quad - \text{crude sum of squares for the mean.}\end{aligned}$$

The first three columns of the anova table are shown in Table 3.1.

3.3 Other treatment groupings

The treatments may be grouped into two or more types for reasons other than that one type consists of controls. The consequences for replication and analysis depend on the reasons for including the different types in the experiment. The following examples illustrate some of the possibilities.

Example 3.2 (Rubber trees) Seven varieties of rubber tree are planted in an experiment to compare yields of the varieties. It happens that the varieties have two visually different leaf-types: three of the varieties have round green leaves while the other four have long serrated grey leaves.

The main purpose of the experment is to compare the seven varieties, so they should be equally replicated. However, it is useful to ask if any differences among the varieties can be explained as differences between the two leaf-types. Thus the analysis should fit the chain of models shown in Figure 3.2, where V_L is the subspace of V_T consisting of vectors which are constant on each leaf-type.

Calculations similar to those in Section 3.2 give the partial anova table shown in Table 3.2.

Example 3.3 (Drugs at different stages of development) A pharmaceuticals company wants to compare six treatments for a certain disease. The initial trial will use healthy volunteers, simply to measure the amount of certain chemicals released into

Figure 3.2: Four models in Example 3.2

source	sum of squares	degrees of freedom
mean	CSS(mean)	1
leaf-types	CSS(leaf-types) − CSS(mean)	1
varieties	CSS(varieties) − CSS(leaf-types)	5
residual	← ..... by subtraction ..... →	
Total	$\sum_\omega y_\omega^2$	$7r$

Table 3.2: First three column of anova table in Example 3.2

the blood two hours after the treatments are administered. Three of the treatments are three different doses of a formulation (coded *A*) that has been under development for some time. The other three are three different doses of a new formulation (coded *B*) that has not been so extensively studied. The main aim of the trial is to compare the doses of formulation *A*; the secondary aim is to compare the new formulation with the old one; and the lowest priority is given to comparing doses of the new formulation.

The main effort in the experiment should go into comparing the doses of the old formulation *A*. Sufficient replication must be used for these three treatments, guided by the principles in Sections 2.9 and 2.12. The company decides to use 12 volunteers for each dose, thus using 36 volunteers. However, it has sufficient resources to use 48 volunteers, so the three doses of formulation *B* are assigned to four volunteers each.

Compared to the more limited design with three doses of *A* and only 36 volunteers, the design with the extra 12 volunteers increases the precision of the estimate of σ^2. It also gives more residual degrees of freedom, and hence more power for detecting differences between the doses of *A*. In addition, it gives some information about doses of *B*.

Variances of estimators of some contrasts are as follows.

between two doses of A	$\frac{2}{12}\sigma^2 = \frac{1}{6}\sigma^2$
between a dose of A and a dose of B	$\left(\frac{1}{12}+\frac{1}{4}\right)\sigma^2 = \frac{1}{3}\sigma^2$
between the average effect of A and the average effect of B	$\frac{1}{9}\left(\frac{3}{12}+\frac{3}{4}\right)\sigma^2 = \frac{1}{9}\sigma^2$
between two doses of B	$\frac{2}{4}\sigma^2 = \frac{1}{2}\sigma^2$

If the company is correct in its judgement that replication 12 is sufficient for the doses of A, then comparisons among these are sufficiently precise, as is the comparison between the average effect of A and the average effect of B. Comparisons between doses of B are less precise, but may yield useful information if there are large differences, which should help the company decide whether to proceed with the development of formulation B.

Now it is sensible to split the sum of squares for treatments into three parts:

(i) the sum of squares for differences between formulations, which is calculated like the sum of squares for leaf-types in Example 3.2;

(ii) the sum of squares for differences between doses of A, which is calculated as in Equation (3.3) under the pretence that all doses of B are a single control;

(iii) the sum of squares for differences between doses of B, which is calculated as in Equation (3.3) under the pretence that all doses of A are a single control.

Define treatment factors F, A and B as follows.

	old formulation			new formulation		
treatment	1	2	3	4	5	6
F	1	1	1	2	2	2
A	1	2	3	0	0	0
B	0	0	0	1	2	3

These define vector spaces V_F, V_A and V_B analogous to V_T and V_C. The relationships between these spaces are shown in Figure 3.3. The expectation model corresponding to each of these is given in Table 3.3.

Example 3.4 (Reducing feed for chickens) An experiment is to be conducted to see if chickens can be fed a slightly inferior diet in the 16 weeks before slaughter without affecting their final weight. The chickens are housed in 40 cages of 20 birds each, and feeds are applied to whole cages. The ten treatments are shown in Table 3.4. Thus the non-control treatments can be grouped into three different methods, each of which has several variants.

coordinate (and parameters)	vector (and subspace)
$\mathbb{E}(Y_\omega) = \kappa$	$\mathbb{E}(\mathbf{Y}) \in V_0$
$\mathbb{E}(Y_\omega) = \begin{cases} \lambda & \text{if } T(\omega) \text{ is old} \\ \mu & \text{if } T(\omega) \text{ is new} \end{cases}$	$\mathbb{E}(\mathbf{Y}) \in V_F$
$\mathbb{E}(Y_\omega) = \begin{cases} \tau_{T(\omega)} & \text{if } T(\omega) \text{ is old} \\ \mu & \text{if } T(\omega) \text{ is new} \end{cases}$	$\mathbb{E}(\mathbf{Y}) \in V_A$
$\mathbb{E}(Y_\omega) = \begin{cases} \lambda & \text{if } T(\omega) \text{ is old} \\ \tau_{T(\omega)} & \text{if } T(\omega) \text{ is new} \end{cases}$	$\mathbb{E}(\mathbf{Y}) \in V_B$
$\mathbb{E}(Y_\omega) = \tau_{T(\omega)}$	$\mathbb{E}(\mathbf{Y}) \in V_T$

Table 3.3: Models for $\mathbb{E}(Y_\omega)$ in Example 3.3

		Treatment
Control		1
Reduce protein content by	5%	2
	10%	3
	15%	4
Change diet to a given cheaper	4 weeks	5
one after	8 weeks	6
	12 weeks	7
Replace 5% of the protein by	*A*	8
an equal volume of roughage	*B*	9
of type	*C*	10

Table 3.4: Treatments in the chicken-feeding experiment in Example 3.4

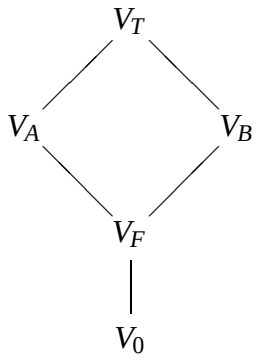

Figure 3.3: Relationships among the subspaces in Example 3.3

The main interest in the experiment is probably in comparing each method with the control treatment, which is the normal feed. Thus Equation (3.1) suggests that we should have r_1 cages with the control and r cages with each method, where $r_1 \approx \sqrt{3}r$. With 40 cages, we could have $r = 8$ and $r_1 = 16$ or $r = 9$ and $r_1 = 13$. All the treatments pertaining to each method should be equally replicated. For the ten treatments in Table 3.4 this suggests replication

3	for all non-control treatments
13	for the control treatment.

However, if we replace one 'quantity of protein' treatment by another 'roughage' treatment, then we might choose replication

3	for the 3 time treatments
4	for the 2 quantity treatments
2	for the 4 roughage treatments
15	for the control treatment.

In practice, such a range of different replications will make the design impossible if there is any blocking (see Chapter 4), so it may be better to opt for equal replication, at least for the non-control treatments.

In this example the sum of squares for treatments should be split into five parts, which can all be calculated following the principles given earlier in this chapter. For generality, suppose that there are n_i treatments pertaining to method i, for $i = 1, 2, 3$:

(i) the sum of squares for the difference between the control treatment and the rest (1 degree of freedom);

(ii) the sum of squares for the differences between methods (2 degrees of freedom);

(iii) the sum of squares for the differences between treatments of the quantity type ($n_1 - 1$ degrees of freedom);

(iv) the sum of squares for the differences between treatments of the time type ($n_2 - 1$ degrees of freedom);

(v) the sum of squares for the differences between treatments of the roughage type ($n_3 - 1$ degrees of freedom).

We shall return to these examples in Chapters 5 and 10.

Questions for Discussion

3.1 Suppose that there are n control treatments, each replicated r_1 times, and m new treatments, each replicated r_t times. Find the optimal ratio r_1/r_t if all treatment comparisons are of interest except those between control treatments.

3.2 Consider the scabbiness experiment in Question 2.1. The seven coded treatments consisted of one 'do nothing' control and six spray treatments, as shown below.

treatment	1	2	3	4	5	6	7
amount of sulphur	0	300	600	1200	300	600	1200
timing	N/A	autumn	autumn	autumn	spring	spring	spring

Give two plausible reasons for the particular choice of unequal replication made in this experiment.

3.3 Reanalyse the data from Question 2.1, taking account of the fact that treatment 1 is a control.