
Chapter 10

The Calculus of Factors

10.1 Introduction

In the preceding chapters we have met all the principles of orthogonal designs (which will be defined in Section 10.11). The plots may be unstructured, as in Chapter 2. There may be a single system of blocks, as in Chapter 4. If there are two systems of blocks then we may have each block of one system meeting each block of the other system, as in Chapter 6 and Section 7.3, or each block of one system may contain several blocks of the other system, as in Section 8.3. The experimental units may be the same as the observational units, or each experimental unit may contain several observational units, as in Section 8.1.

The treatments also may be unstructured, as in Chapter 2. They may be divided into different types, such a control and new treatments, as in Chapter 3. They may consist of all combinations of two or more treatment factors, as in Chapter 5. The different treatment effects may all be estimated in the same stratum, or they may be estimated in two or more strata, as in Sections 8.2–8.4. We may even assume that some interactions are zero, as in Chapter 9.

The purpose of the present chapter is to give a unifying framework that not only encompasses all the designs we have met so far but also permits the construction and analysis of infinitely many more. Once this framework is understood there is absolutely no need to memorize the structure of any individual named design.

The reader may wish to omit proofs on a first reading of this chapter.

10.2 Relations on factors

Consider factors F, G, \dots on a set, which might be either the set Ω of observational units or the set \mathcal{T} of treatments. If the set is Ω then we write $F(\omega)$ for

the level of F which occurs on ω , and the F -class containing plot α is defined to be $\{\omega \in \Omega : F(\omega) = F(\alpha)\}$. We often identify $F(\omega)$ with the F -class containing ω . Likewise, if G is a factor on \mathcal{T} and i is any treatment then $G(i)$ denotes the G -class containing i , that is, the set of all treatments which have the same level of G as i does.

Definition Let F and G be factors on the same set. Then F is *equivalent* to G , or *aliased* with G , if every F -class is also a G -class; that is, F and G are the same apart from the names of their levels.

Notation Write $F \equiv G$ if F is equivalent to G .

Example 10.1 (Example 1.5 continued: Rye-grass) There are two fields, each of which consists of three strips, each of which contains four plots. Thus **field** and **strip** are relevant factors on Ω even before we apply treatments, so we call them *plot factors*. The treatments consist of all combinations of the factors **cultivar** and **nitrogen**. The levels of **cultivar** are Cropper, Melle and Melba, each applied to one whole strip per field. The levels of **nitrogen** are 0, 80, 160 and 240 kg/ha, each applied to one plot per strip. Thus **cultivar** and **nitrogen** are *treatment factors*, as is T itself.

On the data sheet, we may well code the cultivars Cropper, Melle and Melba as 1, 2 and 3 respectively, and abbreviate **cultivar** to C . Thus $C(\omega) = 1$ if and only if the **cultivar** on ω is Cropper. Thus C is merely a renaming of **cultivar** and so $C \equiv \text{cultivar}$.

Aliasing has more uses than simple renaming. If there are two nuisance factors on the plots, it may be beneficial to alias them.

Example 10.2 (Car tyres) There are three nuisance factors in Question 9.4: **car**, **driver** and **day**. If we use a Graeco-Latin square for the design then each of these nuisance factors uses up three degrees of freedom, so there are only three residual degrees of freedom. However, the experimenter is not interested in comparing cars or drivers or days, so this is wasteful. If we allocate each driver to the same car every day then we can use a Latin square for the design and there are six residual degrees of freedom. Now $\text{driver} \equiv \text{car}$ even though drivers and cars are not inherently the same.

If a treatment factor is aliased with a plot factor, it may indicate false replication.

Example 10.3 (Example 1.1 continued: Ladybirds) The field was divided into three areas, from each of which three samples were taken. Thus the plots were the nine samples and **area** was a plot factor. The treatment factor **pesticide** had three levels, each one being applied to a single area. Thus $\text{pesticide} \equiv \text{area}$, which immediately shows that there was no proper replication.

Definition Let F and G be factors on the same set. Then F is *finer* than G , or G is *coarser* than F , if every F -class is contained in a G -class but $F \neq G$.

Notation Write $F \prec G$ if F is finer than G . Write $F \preceq G$ if $F \prec G$ or $F \equiv G$. Also write $G \succ F$ for $F \prec G$ and $G \succeq F$ for $F \preceq G$.

In Example 10.1, $\text{strip} \prec \text{field}$ and $\text{strip} \prec \text{cultivar}$. Also $T \prec \text{cultivar}$ and $T \prec \text{nitrogen}$.

There are two special factors on Ω (indeed, on any set).

the universal factor U This is defined by $U(\omega) = \Omega$ for all ω in Ω . Thus U has a single class, which is the whole of Ω , or the *universe*. It is the **uncaring** factor, because it makes no distinctions between units.

the equality factor E This is defined by $E(\omega) = \omega$ for all ω in Ω . Thus E has as many classes as there are plots. There is a class for **each** and **every** plot. If α and β are in the same E -class then $\alpha = \beta$: they are **equal**.

In Example 10.1, $E = \text{plot}$ and $\text{plot} \prec \text{strip} \prec \text{field}$. In Example 10.2, $E = \text{car} \wedge \text{day}$. In Example 10.3, $E = \text{sample}$ and $\text{sample} \prec \text{area}$. In Example 8.1, $E = \text{calf}$ and $\text{calf} \prec \text{pen}$. In Example 6.1, $E = \text{judge} \wedge \text{tasting position}$.

For every factor F , we have

$$E \preceq F \preceq U. \tag{10.1}$$

10.3 Operations on factors

We have already met the idea of $F \wedge G$, where F and G are factors on the same set. The levels of $F \wedge G$ (which may be pronounced ‘ F down G ’) are the combinations of levels of F and G . Formally we define:

Definition The *infimum* of factors F and G on the same set is the factor $F \wedge G$ whose classes are the non-empty intersections of F -classes with G -classes.

Thus, on Ω , the $F \wedge G$ -class containing the plot α is

$$\{\omega \in \Omega : F(\omega) = F(\alpha) \text{ and } G(\omega) = G(\alpha)\}.$$

Example 10.1 revisited (Rye-grass) Within each field, each cultivar occurs on one whole strip. Therefore $\text{field} \wedge \text{cultivar} = \text{strip}$.

The factor $F \wedge G$ is called the infimum of F and G because it satisfies the following two conditions:

- (i) $F \wedge G \preceq F$ and $F \wedge G \preceq G$;
- (ii) if H is a factor such that $H \preceq F$ and $H \preceq G$ then $H \preceq F \wedge G$.

This concept should already be familiar from the natural numbers, with ‘is finer than’ replaced by ‘divides’. The *highest common factor* h of two natural numbers n and m satisfies

- (i) h divides n and h divides m ;
- (ii) if k is a natural number such that k divides n and k divides m then k divides h .

There is a dual concept, that of the *least common multiple*. Now, l is the least common multiple of two natural numbers n and m if

- (i) n divides l and m divides l ;
- (ii) if k is a natural number such that n divides k and m divides k then l divides k .

The dual concept also occurs for factors on a set, defined very like the least common multiple but with ‘divides’ replaced by ‘is finer than’.

Definition The *supremum* of factors F and G on the same set is the unique factor (up to equivalence) $F \vee G$ which satisfies

- (i) $F \preceq F \vee G$ and $G \preceq F \vee G$;
- (ii) if H is a factor such that $F \preceq H$ and $G \preceq H$ then $F \vee G \preceq H$.

The supremum $F \vee G$ may be pronounced ‘ F up G ’.

Unfortunately, the recipe for writing down the class of $F \vee G$ containing ω is not so simple as it was for $F \wedge G$. Starting with ω , write down all the plots in the same F -class as ω . Then write down all the plots which are in the same G -class as any plot written down so far. Then write down all the plots which are in the same F -class as any plot written down so far. Continue alternating between F and G until no new plots are added. Then the set of plots which have been written down forms the $F \vee G$ -class containing ω .

In other words, α and β are in the same class of $F \vee G$ if there is a finite sequence of elements (plots if we are dealing with Ω , treatments if we are dealing with \mathcal{T})

$$\alpha = \alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_n, \beta_n = \beta$$

such that

$$\alpha_i \text{ and } \beta_i \text{ have the same level of } F \text{ for } i = 1, 2, \dots, n$$

and

$$\beta_j \text{ and } \alpha_{j+1} \text{ have the same level of } G \text{ for } j = 1, \dots, n-1.$$

In a complete-block design, every block contains a plot with each treatment, so the first two steps of this process show that $B \vee T = U$, where B is the block factor and T the treatment factor.

Similarly, in Example 6.1, judge \vee tasting position = U .

Example 10.1 revisited (Rye-grass) The plan for the experiment is shown again in Figure 10.1, this time with the plots numbered. We shall calculate $\text{strip} \vee T$.

Start at plot number 16. The plots in the same strip are plots 13, 14, 15 and 16. The plots with the same treatment as plot 13 are plots 5 and 13; the plots with the same treatment as plot 14 are plots 7 and 14; the plots with the same treatment as plot 15 are plots 8 and 15; the plots with the same treatment as plot 16 are plots 6 and 16. So far we have written down $\{5, 6, 7, 8, 13, 14, 15, 16\}$. These form two whole strips, and so the process stops. These plots are precisely those that have the cultivar Melba: in other words,

$$(\text{strip} \vee T)(16) = \text{cultivar}(16).$$

A similar argument applies no matter what plot we start with, and therefore $\text{strip} \vee T = \text{cultivar}$.

Strip 1	Strip 2	Strip 3	Strip 4	Strip 5	Strip 6
0 1	160 5	240 9	160 13	80 17	0 21
160 2	80 6	80 10	0 14	160 18	80 22
80 3	0 7	160 11	240 15	0 19	240 23
240 4	240 8	0 12	80 16	240 20	160 24
↑	↑	↑	↑	↑	↑
Cropper	Melba	Melle	Melba	Cropper	Mella

Figure 10.1: Layout of the experiment in Example 10.1

Another way to calculate $F \vee G$ is to draw a rectangular diagram like those in Tables 1.5–1.6. The rows and columns represent the levels of F and G (in either order), and a tick in a cell indicates that there is at least one element which has the given levels of F and G . Here an ‘element’ is a plot if the set is Ω and a treatment if the set is \mathcal{T} . Thus the cells with ticks are the classes of $F \wedge G$. Starting at any element, you may move to any other cell in the same row that has a tick in it; then you may move within the column to any other cell with a tick in it; and so on. All the elements which can be reached in this way make up the class of $F \vee G$ which contains the starting element.

Example 10.1 revisited (Rye-grass) The information in Figure 10.1 is presented in rectangular form in Figure 10.2. This shows that $\text{strip} \vee T = \text{cultivar}$ and $\text{strip} \wedge T = E$.

Strip	Treatment											
	Cropper				Melle				Melba			
	0	80	160	240	0	80	160	240	0	80	160	240
1	✓ 1	✓ 3	✓ 2	✓ 4								
2									✓ 7	✓ 6	✓ 5	✓ 8
3					✓ 12	✓ 10	✓ 11	✓ 9				
4									✓ 14	✓ 16	✓ 13	✓ 15
5	✓ 19	✓ 17	✓ 18	✓ 20								
6					✓ 21	✓ 22	✓ 24	✓ 23				

Figure 10.2: A different presentation of the information in Figure 10.1

		G-classes							
					✓	✓			
					✓	✓		✓	
F-classes	✓	✓	✓			✓			✓
	✓		✓			✓		✓	✓
	✓	✓	✓			✓		✓	✓

Figure 10.3: A hypothetical example where the calculation of $F \vee G$ requires more than two steps

The process of finding all the elements in a class of $F \vee G$ does not always stop after two steps. Figure 10.3 shows a hypothetical example in which $F \wedge G$ has 21 classes and $F \vee G$ has 2 classes.

Consider a factorial experiment with treatment factors F , G and H . Inside each F -class, all combinations of levels of G and H occur. This shows that $(F \wedge G) \vee (F \wedge H) = F$.

Example 10.4 (Nematodes) The experiment in Question 4.1 has nine treatments. One is a control: no fumigant is applied. Each of the others consists of one of four types of fumigant (coded S , K , M and N) in either a single or a double dose. The best way to think about this is to consider the factor **type** to have five levels— S , K , M , N and none, which was coded Z in Question 4.1—and the factor **dose** to have the three levels none, single and double, which were coded 0, 1 and 2. Thus the nine treatments have the structure shown in Figure 10.4.

This figure shows that **dose** \vee **type** has two classes. One contains just the control treatment (no fumigant) while the other contains all the other treatments. It is convenient to name this factor **fumigant**, as in Table 10.1 and Question 4.1. Thus

dose	type				
	Z	S	K	M	N
none	✓				
single		✓	✓	✓	✓
double		✓	✓	✓	✓

Figure 10.4: The nine treatments in Example 10.4

we have $\text{dose} \vee \text{type} = \text{fumigant}$.

treatment	1	2	3	4	5	6	7	8	9
dose	0	1	2	1	2	1	2	1	2
type	Z	S	S	K	K	M	M	N	N
fumigant	1	2	2	2	2	2	2	2	2

Table 10.1: Treatment factors in Example 10.4

Note that if $F \preceq G$ then $F \wedge G = F$ and $F \vee G = G$.

Definition A factor is *uniform* if all of its classes have the same size.

Notation If factor F is uniform then k_F denotes the size of all its classes.

Thus if F is a uniform factor on Ω then $n_F k_F = N$.

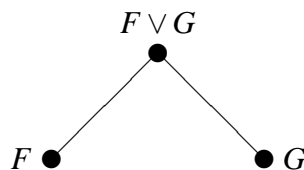
The factors U and E are always uniform, with $k_E = 1$ and k_U being the size of the set. On the other hand, $n_U = 1$ and n_E is the size of the set.

In Example 10.1, field, strip, plot, cultivar and nitrogen are all uniform. In Example 10.4, none of type, dose and fumigant is uniform on \mathcal{T} .

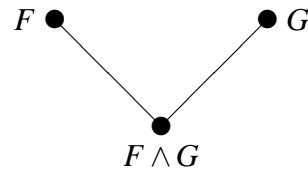
10.4 Hasse diagrams

It is convenient to show the relationships between factors by drawing Hasse diagrams, which are named after the German mathematician Hasse. Draw a dot for each factor. If $F \prec G$ then draw the dot for G (roughly) above the dot for F and join F to G with a line (which may go through other dots). The dot for U is always at the top. If E is included, it is at the bottom.

If neither of F and G is finer than the other, make sure that the diagram contains a dot for $F \vee G$, as follows.

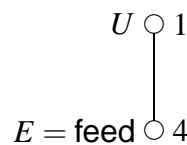
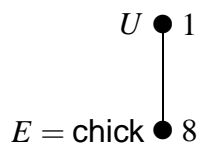


If F and G are plot factors and neither is finer than the other, make sure that the diagram contains a dot for $F \wedge G$, as follows.



It is usually best to draw two separate Hasse diagrams, one for the plot factors and one for the treatment factors. I distinguish them by using a filled dot for a plot factor and an open dot for a treatment factor. Beside the dot for factor F , write its number of levels, n_F .

If the set is unstructured then the Hasse diagram is very simple: it contains just U and E and the line joining them. For example, the Hasse diagram for the plot factors in Example 5.10 is in Figure 10.5(a) and the Hasse diagram for the treatment factors in Example 8.1 is in Figure 10.5(b).



(a) Plot structure in Example 5.10

(b) Treatment structure in Example 8.1

Figure 10.5: Hasse diagrams for two unstructured sets

If there is only one factor apart from U and E then the three factors form a chain. The plot factors for Examples 1.9 and 8.1 are shown in Figure 10.6(a) and (b). Figure 10.6(c) shows the Hasse diagram for the treatment factors if there are eight treatments, one of which is a control. Note that there is nothing in the Hasse diagram to indicate whether a factor is uniform or not. If there were eight treatments divided into two types, with four of each type, then the Hasse diagram would still look like Figure 10.6(c).

If there are two factors other than U and E and neither is finer than the other then we obtain the diamond shape in Figure 10.6(d) and (e). If one is finer than the other then we have a chain of four factors, as in Figure 10.6(f). The treatment structure in Example 10.4 is more complicated, and is shown in Figure 10.6(g). We shall meet further types of Hasse diagram later in the chapter.

10.5 Subspaces defined by factors

Let F be a factor. As we have done in particular cases in Sections 2.3, 3.2–3.3, 4.2, 5.1, 6.4, and 8.1.3, we define V_F to be the set of vectors which are constant on each level of F . Then $\dim V_F = n_F$. In particular, V_U is the space that we have previously called V_0 , with dimension 1, and V_E is the whole space.

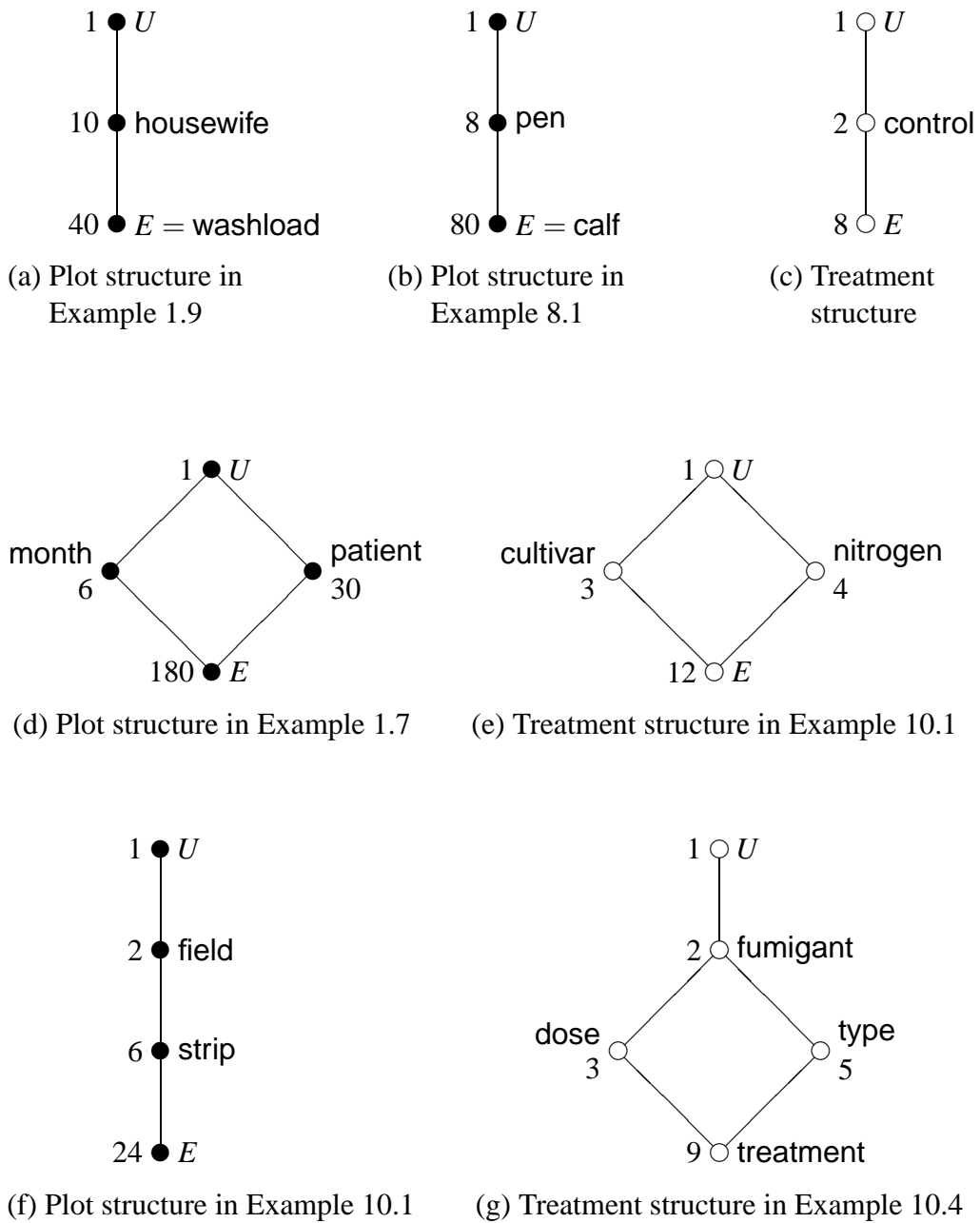


Figure 10.6: Hasse diagrams for several examples

Proposition 10.1 *If F and G are factors and $F \preceq G$ then $V_G \subseteq V_F$.*

Proof This is just like the proof of Lemma 8.1. ■

Proposition 10.2 *If F and G are factors then $V_F \cap V_G = V_{F \vee G}$.*

Proof We have $F \preceq F \vee G$, so Proposition 10.1 shows that $V_{F \vee G} \subseteq V_F$. Similarly, $V_{F \vee G} \subseteq V_G$. Therefore $V_{F \vee G} \subseteq V_F \cap V_G$.

Conversely, we need to show that $V_F \cap V_G \subseteq V_{F \vee G}$. Suppose that \mathbf{v} is a vector in $V_F \cap V_G$. We can turn \mathbf{v} into a factor H by putting $H(\omega) = v_\omega$. Since $\mathbf{v} \in V_F$, we know that \mathbf{v} is constant on each class of F , which implies that each class of F is contained in a single class of H . In other words, $F \preceq H$. Similarly, $G \preceq H$. By the definition of supremum, $F \vee G \preceq H$. By Proposition 10.1, $V_H \subseteq V_{F \vee G}$. However, it is clear that $\mathbf{v} \in V_H$, and so $\mathbf{v} \in V_{F \vee G}$. Thus every vector in $V_F \cap V_G$ is in $V_{F \vee G}$, and so $V_F \cap V_G \subseteq V_{F \vee G}$.

Hence $V_F \cap V_G = V_{F \vee G}$. ■

If F and G are treatment factors then we want to consider both V_F and V_G to be expectation models. By the Intersection Principle, we should also consider $V_F \cap V_G$ to be an expectation model. Proposition 10.2 shows that this intersection is just $V_{F \vee G}$. Therefore we insist that $F \vee G$ must be a treatment factor if F and G are.

The Orthogonality Principle says that if V_F and V_G are both expectation models then $V_F \cap (V_F \cap V_G)^\perp$ should be orthogonal to $V_G \cap (V_F \cap V_G)^\perp$. Since $V_F \cap V_G = V_{F \vee G}$, this motivates the following definition.

Definition Factors F and G are *orthogonal* to each other if the subspace $V_F \cap V_{F \vee G}^\perp$ is orthogonal to the subspace $V_G \cap V_{F \vee G}^\perp$.

Note that if $F \preceq G$ then $F \vee G = G$ and so

$$V_G \cap V_{F \vee G}^\perp = V_G \cap V_G^\perp = \{\mathbf{0}\},$$

which is orthogonal to all vectors, so F is orthogonal to G according to our definition.

This definition is consistent with the definition of orthogonal block design in Section 4.2. So long as $B \vee T = U$, a block design is orthogonal precisely when the treatment factor T is orthogonal to the block factor B .

Our definition of orthogonality is convenient for verifying the Orthogonality Principle but fairly useless for checking whether two factors are orthogonal to each other. We now show that orthogonality is equivalent to two more tractable conditions.

Theorem 10.3 *Factors F and G are orthogonal to each other if and only if $P_{V_F} P_{V_G} = P_{V_G} P_{V_F}$. If they are orthogonal to each other then $P_{V_F} P_{V_G} = P_{V_{F \vee G}}$.*

Proof Put $W_F = V_F \cap V_{F \vee G}^\perp$, $W_G = V_G \cap V_{F \vee G}^\perp$ and $W = (V_F + V_G)^\perp$. If F is orthogonal to G then the whole space is the orthogonal direct sum $V_{F \vee G} \oplus W_F \oplus W_G \oplus W$. Thus any vector \mathbf{v} has a unique expression as $\mathbf{v} = \mathbf{x} + \mathbf{v}_F + \mathbf{v}_G + \mathbf{w}$ where $\mathbf{x} \in V_{F \vee G}$, $\mathbf{v}_F \in W_F$, $\mathbf{v}_G \in W_G$ and $\mathbf{w} \in W$. Now $P_{V_G} \mathbf{v} = \mathbf{x} + \mathbf{v}_G$, because \mathbf{x} and \mathbf{v}_G are in V_G while \mathbf{v}_F and \mathbf{w} are orthogonal to V_G . Hence $P_{V_F}(P_{V_G} \mathbf{v}) = P_{V_F}(\mathbf{x} + \mathbf{v}_G) = \mathbf{x}$. Similarly, $P_{V_G}(P_{V_F} \mathbf{v}) = P_{V_G}(\mathbf{x} + \mathbf{v}_F) = \mathbf{x}$. This is true for all \mathbf{v} , so $P_{V_F}P_{V_G} = P_{V_G}P_{V_F}$. Moreover, $V_{F \vee G}^\perp = W_F \oplus W_G \oplus W$, so $P_{V_{F \vee G}} \mathbf{v} = \mathbf{x}$ and therefore $P_{V_F}P_{V_G} = P_{V_{F \vee G}}$.

Conversely, suppose that $P_{V_F}P_{V_G} = P_{V_G}P_{V_F}$. Then $P_{V_F}P_{V_G} \mathbf{v} \in V_F \cap V_G = V_{F \vee G}$ for all vectors \mathbf{v} . In particular, if $\mathbf{v} \in W_G$ then $\mathbf{v} \in V_G$ so $P_{V_G} \mathbf{v} = \mathbf{v}$ and so $P_{V_F} \mathbf{v} = P_{V_F}P_{V_G} \mathbf{v} \in V_{F \vee G}$, which is orthogonal to \mathbf{v} . The only way that a projection of a vector can be orthogonal to that vector is that the projection is zero, so $P_{V_F} \mathbf{v} = \mathbf{0}$. In other words, \mathbf{v} is orthogonal to V_F . Thus W_G is orthogonal to V_F , which implies that F is orthogonal to G . ■

Corollary 10.4 *Suppose that factors F , G and H are pairwise orthogonal. Then $F \vee G$ is orthogonal to H .*

Proof Because F is orthogonal to G , $P_{V_{F \vee G}} = P_{V_F}P_{V_G}$. Because H is orthogonal to both F and G , P_{V_H} commutes with both P_{V_F} and P_{V_G} . Therefore

$$P_{V_{F \vee G}}P_{V_H} = P_{V_F}P_{V_G}P_{V_H} = P_{V_F}P_{V_H}P_{V_G} = P_{V_H}P_{V_F}P_{V_G} = P_{V_H}P_{V_{F \vee G}}$$

and so $F \vee G$ is orthogonal to H . ■

Theorem 10.5 *Factors F and G are orthogonal to each other if and only if the following two conditions are satisfied within each class of $F \vee G$ separately:*

- (i) every F -class meets every G -class;
- (ii) all these intersections have size proportional to the product of the sizes of the relevant F -class and G -class.

Proof Let p_i be the size of the i -th class of F , q_j the size of the j -th class of G , and s_{ij} the size of their intersection. Then s_{ij} must be zero whenever the i -th class of F and the j -th class of G are in different classes of $F \vee G$. Hence conditions (i) and (ii) together are equivalent to the existence of constants c_Δ , for each class Δ of $F \vee G$, such that

$$s_{ij} = c_\Delta p_i q_j \tag{10.2}$$

whenever the i -th class of F and the j -th class of G are in Δ . (A counting argument shows that c_Δ must be the reciprocal of the size of Δ , but we do not need to use that.)

Let \mathbf{y} be the data vector. We showed in Section 2.6 that the coordinate of $P_{V_F} \mathbf{y}$ is equal to $\sum_{F=i} / p_i$ on every element in the i -th class of F . Hence the coordinate of $P_{V_G}P_{V_F} \mathbf{y}$ is equal to

$$\left(\sum_i s_{ij} \frac{\sum_{F=i}}{p_i} \right) / q_j \tag{10.3}$$

on every element in the j -th class of G . If F is orthogonal to G then $P_{V_F}P_{V_G} = P_{V_{F \vee G}}$ and so expression (10.3) has the same value for all j such that the j -th class of G is in a given class Δ of $F \vee G$. This is true no matter what the data \mathbf{y} are, so s_{ij}/q_j is constant for all such j . Similarly, s_{ij}/p_i is constant for all i such that the i -th class of F is in Δ . Therefore Equation (10.2) is true.

Conversely, if Equation (10.2) holds then expression (10.3) is equal to

$$c_{\Delta} \sum_i' \text{sum}_{F=i} = c_{\Delta} \text{sum}_{F \vee G = \Delta},$$

where \sum' denotes the sum over those i for which the i -th class of F is in Δ . Similarly, the coordinate of $P_{V_F}P_{V_G}\mathbf{y}$ is equal to

$$\left(\sum_j s_{ij} \frac{\text{sum}_{G=j}}{q_j} \right) / p_i = c_{\Delta} \sum_j' \text{sum}_{G=j} = c_{\Delta} \text{sum}_{F \vee G = \Delta}$$

on every element of Δ . Hence $P_{V_F}P_{V_G} = P_{V_G}P_{V_F}$ and so F is orthogonal to G . ■

Note that condition (i) of Theorem 10.5 implies that the process of calculating $F \vee G$ is complete in two steps. Condition (ii) is most often achieved by having all classes of $F \wedge G$ of the same size within each class of $F \vee G$, but possibly different sizes in different classes of $F \vee G$.

For treatment factors, the property of orthogonality may be different on the treatment set \mathcal{T} from what it is on the set Ω of plots. This is an important point that we shall return to in Section 10.11.

Example 10.4 revisited (Nematodes) On \mathcal{T} , the factors **dose** and **type** are orthogonal to each other.

On Ω , if the control treatment is replicated r_1 times and all other treatments are replicated r_2 times, then **dose** is orthogonal to **type**. Figure 10.7(a) shows the replication in the experiment in Question 4.1: this makes **dose** orthogonal to **type**. Some other patterns of replication that also lead to orthogonality are shown in Figure 10.7(b)–(d).

10.6 Orthogonal decomposition

As we have previously done in particular cases, we now want to define a W -subspace associated with each factor in such a way that W_F is orthogonal to W_G if F and G are different factors. If $F \prec G$ then $V_G \subset V_F$ so we want W_F to be contained in V_F but orthogonal to V_G . This is the case for all G coarser than F , so we want W_F to be orthogonal to the space $\sum_{G \succ F} V_G$. It turns out that the Intersection Principle and the Orthogonality Principle give us just the right conditions to make the W -subspaces an orthogonal decomposition of the whole space.

dose	type				
	Z	S	K	M	N
none	16				
single		4	4	4	4
double		4	4	4	4

(a)

dose	type				
	Z	S	K	M	N
none	16				
single		4	5	3	4
double		4	5	3	4

(b)

dose	type				
	Z	S	K	M	N
none	16				
single		5	5	5	5
double		3	3	3	3

(c)

dose	type				
	Z	S	K	M	N
none	16				
single		3	6	9	6
double		1	2	3	2

(d)

Figure 10.7: Several different patterns of replication in Example 10.4 that make dose orthogonal to type

Theorem 10.6 *Let \mathcal{F} be a set of inequivalent factors on the same set. Suppose that \mathcal{F} satisfies*

- (a) *if $F \in \mathcal{F}$ and $G \in \mathcal{F}$ then $F \vee G \in \mathcal{F}$;*
- (b) *if $F \in \mathcal{F}$ and $G \in \mathcal{F}$ then F is orthogonal to G .*

Define subspaces W_F for F in \mathcal{F} by

$$W_F = V_F \cap \left(\sum_{G \succ F} V_G \right)^\perp$$

and put $d_F = \dim W_F$. Then

- (i) *if F and G are different factors in \mathcal{F} then W_F is orthogonal to W_G ;*
- (ii) *if $F \in \mathcal{F}$ then V_F is the orthogonal direct sum of those W_G for which $G \succ F$: in particular, if $E \in \mathcal{F}$ then the whole space is equal to*

$$\bigoplus_{F \in \mathcal{F}} W_F;$$

- (iii) *if $F \in \mathcal{F}$ then*

$$d_F = n_F - \sum_{G \succ F} d_G. \tag{10.4}$$

Proof (i) If $F \prec G$ then $V_G \subseteq \sum_{H \succ F} V_H$ and so the definition of W_F shows that $W_F \subseteq V_F \cap V_G^\perp$. Thus $W_F \subseteq V_G^\perp$ while $W_G \subseteq V_G$, and so W_F is orthogonal to W_G .

A similar argument holds if $G \prec F$.

If neither of F and G is finer than the other then $F \vee G$ is different from both F and G and so $F \vee G \succ F$ and $F \vee G \succ G$. Therefore $W_F \subseteq V_F \cap V_{F \vee G}^\perp$ and $W_G \subseteq V_G \cap V_{F \vee G}^\perp$. Since the factors F and G are orthogonal to each other, the subspaces $V_F \cap V_{F \vee G}^\perp$ and $V_G \cap V_{F \vee G}^\perp$ are orthogonal to each other. Therefore W_F is orthogonal to W_G .

(ii) We use induction on F . If there is no G in \mathcal{F} such that $G \succ F$ then $W_F = V_F$ and the result is true.

Suppose that the result is true for all factors G in \mathcal{F} for which $G \succ F$. Then $\sum_{G \succ F} V_G$ is the orthogonal direct sum of the spaces W_G for which $G \succ F$. The definition of W_F shows that V_F is the orthogonal direct sum of W_F and $\sum_{G \succ F} V_G$. Hence

$$V_F = \bigoplus_{G \succ F} W_G. \quad (10.5)$$

(iii) The dimension of the right-hand side of Equation (10.5) is equal to $\sum_{G \succ F} d_F$, which is equal to $d_F + \sum_{G \succ F} d_G$. The dimension of the left-hand side is equal to n_F . Therefore $n_F = d_F + \sum_{G \succ F} d_G$, and Equation (10.4) follows. ■

Note that

$$\left(\sum_{G \succ F} V_G \right)^\perp = \bigcap_{G \succ F} V_G^\perp,$$

and so

$$W_F = V_F \cap \bigcap_{G \succ F} V_G^\perp.$$

The number d_F is called the number of *degrees of freedom for F*. It is important to realize that, although n_F depends only on F , d_F depends also on what other factors are included in \mathcal{F} .

Let \mathbf{y} be the data vector and F a factor. Then $P_{V_F} \mathbf{y}$ is the orthogonal projection of \mathbf{y} onto V_F : it is called the *fit for F*. The coordinate of $P_{V_F} \mathbf{y}$ on ω is the mean of the values of \mathbf{y} on the F -class which contains ω , which is equal to

$$\frac{\text{total of } \mathbf{y} \text{ on the } F\text{-class containing } \omega}{\text{size of the } F\text{-class containing } \omega}.$$

Sometimes the vector of fitted values for F is displayed as a *table of means*.

As usual, the *crude sum of squares for F* is defined to be $\|P_{V_F} \mathbf{y}\|^2$. Thus

$$\text{CSS}(F) = \sum_{F\text{-classes}} \frac{(\text{total of } \mathbf{y} \text{ on the } F\text{-class})^2}{\text{size of the } F\text{-class}}.$$

The fit, means and crude sum of squares for F depend only on the data \mathbf{y} and the factor F . In general, effects and sums of squares depend also on which other factors are being considered, so we shall define them only when the hypotheses of Theorem 10.6 are satisfied.

Theorem 10.7 *If \mathcal{F} is a set of inequivalent factors which satisfies the hypotheses of Theorem 10.6, then*

(i) *if F and G are different factors in \mathcal{F} then $P_{W_F}\mathbf{y}$ is orthogonal to $P_{W_G}\mathbf{y}$;*

(ii) *if $F \in \mathcal{F}$ then $P_{V_F}\mathbf{y} = \sum_{G \succ F} P_{W_G}\mathbf{y}$; in particular, if $E \in \mathcal{F}$ then*

$$\mathbf{y} = P_V\mathbf{y} = P_{V_E}\mathbf{y} = \sum_{F \in \mathcal{F}} P_{W_F}\mathbf{y};$$

(iii) *if $F \in \mathcal{F}$ then $\|P_{V_F}\mathbf{y}\|^2 = \sum_{G \succ F} \|P_{W_G}\mathbf{y}\|^2$.*

Proof (i) Theorem 10.6(i) shows that W_F is orthogonal to W_G , and so every vector in W_F is orthogonal to every vector in W_G .

(ii) Theorem 10.6(ii) shows that V_F is the direct sum of the spaces W_G for which $G \succ F$. Hence the projection of \mathbf{y} onto V_F is the sum of the projections of \mathbf{y} onto those W_G .

(iii) Since the vectors $P_{W_G}\mathbf{y}$ with $G \succ F$ are mutually orthogonal and sum to $P_{V_F}\mathbf{y}$, the result follows from Pythagoras' Theorem.

When the conditions of Theorem 10.6 are satisfied then we call $P_{W_F}\mathbf{y}$ the *effect* of F and $\|P_{W_F}\mathbf{y}\|^2$ the *sum of squares* for F . Now

$$\text{fit for } F = \text{effect of } F + \sum_{G \succ F} \text{effect of } G,$$

and so

$$\text{effect of } F = \text{fit for } F - \sum_{G \succ F} \text{effect of } G. \quad (10.6)$$

Similarly,

$$\text{CSS}(F) = \|P_{V_F}\mathbf{y}\|^2 = \text{SS}(F) + \sum_{G \succ F} \text{SS}(G),$$

and so

$$\text{SS}(F) = \text{CSS}(F) - \sum_{G \succ F} \text{SS}(G). \quad (10.7)$$

All the analyses which we have looked at so far use special cases of Theorems 10.6 and 10.7.

treatment	old formulation			new formulation		
	1	2	3	4	5	6
<i>F</i>	1	1	1	2	2	2
<i>A</i>	1	2	3	0	0	0
<i>B</i>	0	0	0	1	2	3

Table 10.2: Treatment factors in Example 10.5

10.7 Calculations on the Hasse diagram

It is very convenient to calculate degrees of freedom by using the Hasse diagram. Start at the top. There is no factor coarser than U , so Equation (10.4) shows that $d_U = n_U = 1$. Then work down the Hasse diagram, using Equation (10.4) at each dot F to calculate d_F from n_F and the degrees of freedom for those dots above F . At each dot, write d_F beside n_F .

When doing this process by hand, you can use one colour for the numbers of levels n_F and another for the degrees of freedom d_F . In this book I adopt the convention that d_F is always shown either immediately to the right of n_F or immediately below n_F .

Figures 10.5 and 10.6 are redisplayed as Figures 10.8 and 10.9 with the degrees of freedom included.



(a) Plot structure in Example 5.10 (b) Treatment structure in Example 8.1

Figure 10.8: Hasse diagrams for two unstructured sets, showing numbers of levels and degrees of freedom

Example 10.5 (Example 3.3 continued: Drugs at different stages of development)

The treatments consist of three doses of the old formulation A and three doses (not comparable with the first three) of a new formulation B . Table 10.2 repeats the information from Chapter 3, showing a factor F which distinguishes between the two formulations, and factors A and B which are designed for testing for differences among the different doses of each formulation. These give the Hasse diagram in Figure 10.10(a). Notice that $d_E = 0$; that is, the space W_E consists of the zero vector only. Thus there are only three treatment sums of squares in addition to the mean. This agrees with the finding in Chapter 3.

Example 10.6 (Rats) Six different diets were fed to 60 rats, 10 rats per diet. The rats were weighed at the beginning and the end of the experiment, and their weight

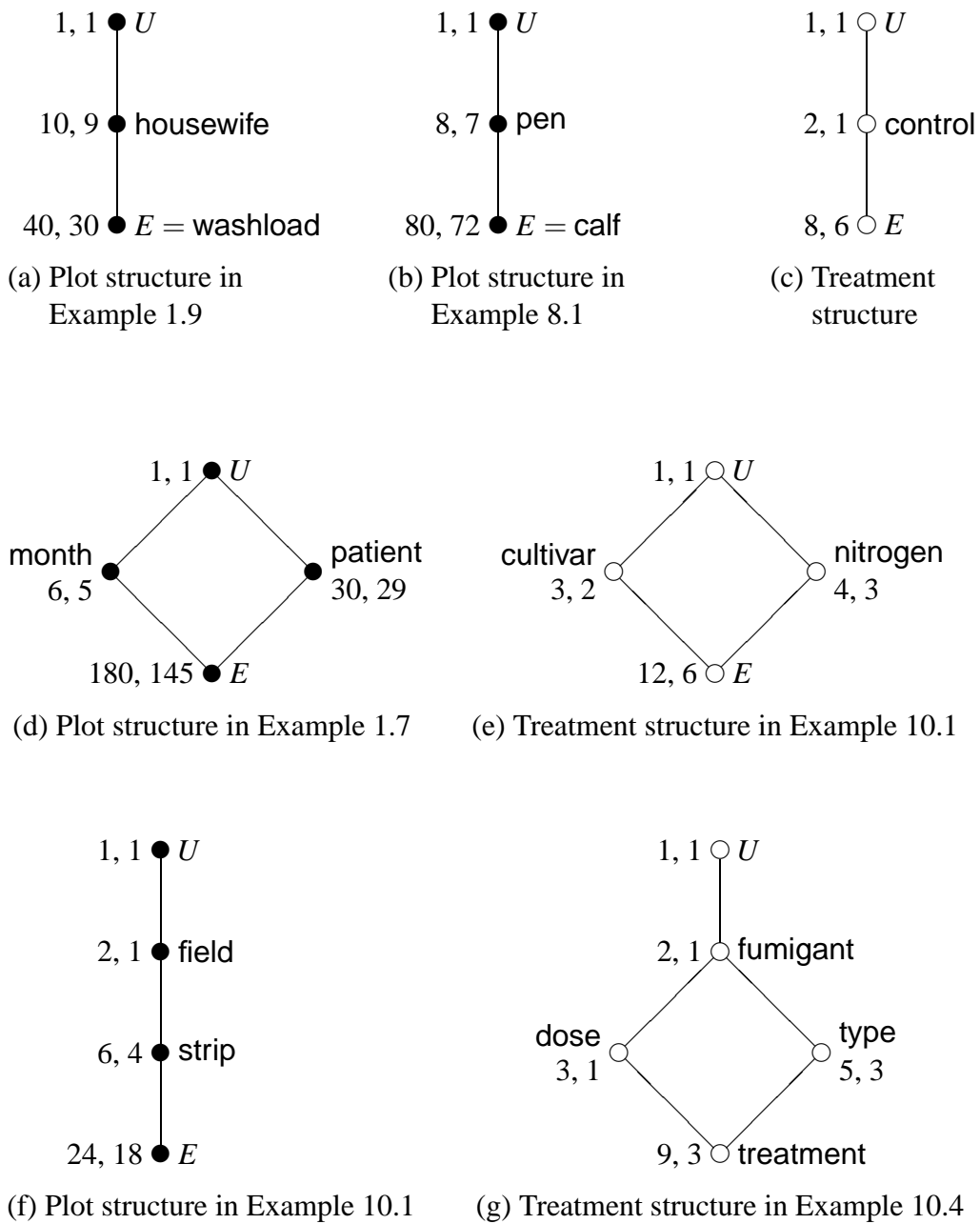


Figure 10.9: Hasse diagrams for several examples, showing numbers of levels and degrees of freedom

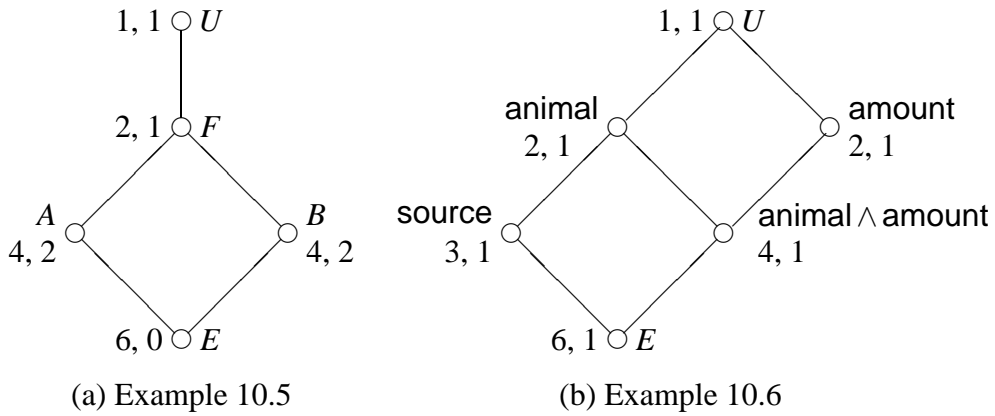


Figure 10.10: Hasse diagrams for two treatment structures

treatment	1	2	3	4	5	6
source	beef		pork		cereal	
amount	low	high	low	high	low	high
animal	1	1	1	1	2	2
animal \wedge amount	1	2	1	2	3	4

Table 10.3: Treatment factors in Example 10.6

gain was recorded. The six diets consisted of three sources of protein, each at a high or low amount. The sources of protein were beef, pork and cereal, so there is a relevant factor **animal** which distinguishes between the two animal sources and the cereal. The treatment factors are displayed in Table 10.3 and Figure 10.10(b).

If the treatments consist of all combinations of the levels of three treatment factors F , G and H then we obtain the Hasse diagram in Figure 10.11.

Although it is impractical to write effects or sums of squares on the Hasse diagram, the same method of calculation gives both of these. The fit for each factor is easy to calculate. Then work down the Hasse diagram, using Equation (10.6) at each point F to calculate the effect of F from the fit for F and the effects for those factors above F . The crude sum of squares for each factor is also very easy to calculate from its table of means or table of totals. Then work down the Hasse diagram, using Equation (10.7) at each point F to calculate the sum of squares for F from the crude sum of squares for F and the sums of squares for those factors above F .

10.8 Expectation models

In Chapter 4 we saw that if the plots are structured then V_0 needs to be removed from both the treatment subspace and the block subspace. In other words, the universal factor U must be considered both a treatment factor and a plot factor.

This consideration, together with Theorem 10.6, motivates the following definition.

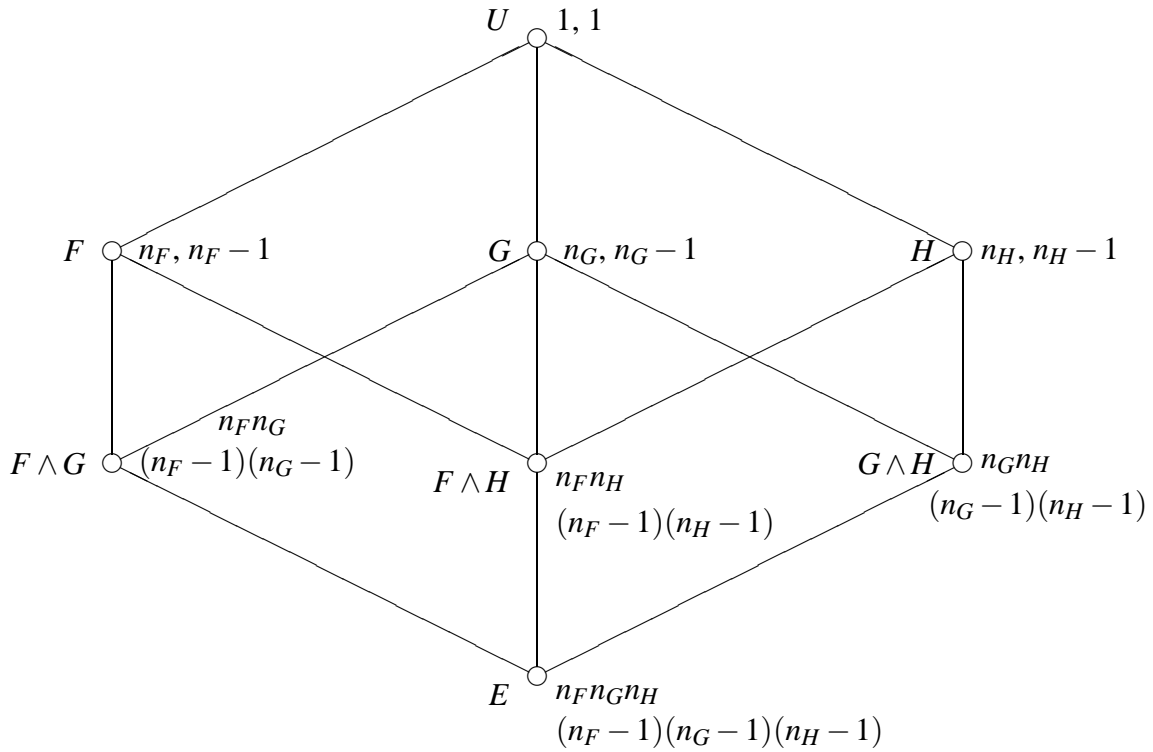


Figure 10.11: Hasse diagram for a factorial treatment structure with three factors F , G and H

Definition A set \mathcal{G} of inequivalent factors on \mathcal{T} is an *orthogonal treatment structure* if

- (i) $U \in \mathcal{G}$;
- (ii) if $F \in \mathcal{G}$ and $G \in \mathcal{G}$ then $F \vee G \in \mathcal{G}$;
- (iii) if $F \in \mathcal{G}$ and $G \in \mathcal{G}$ then F is orthogonal to G .

For the rest of this section we assume that \mathcal{G} is an orthogonal treatment structure.

For each F in \mathcal{G} , we want to consider V_F as an expectation model. By the Sum Principle, if F and G are both in \mathcal{G} then $V_F + V_G$ should also be an expectation model. Thus the expectation models should be all sums of zero, one, two or more of the spaces V_F for F in \mathcal{G} . Theorem 10.6 shows that each expectation model is the direct sum of some of the spaces W_F with F in \mathcal{G} .

Theorem 10.8 Let \mathcal{H} be a subset of \mathcal{G} . Then $\bigoplus_{H \in \mathcal{H}} W_H$ is an expectation model if and only if \mathcal{H} satisfies the following condition:

$$\text{if } F \in \mathcal{H} \text{ and } G \in \mathcal{G} \text{ and } F \preceq G \text{ then } G \in \mathcal{H}. \tag{10.8}$$

Proof Put $M = \bigoplus_{H \in \mathcal{H}} W_H$. First suppose that M is an expectation model. If $F \in \mathcal{H}$ then $W_F \subseteq M$ so there is some H in \mathcal{G} such that $W_F \subseteq V_H \subseteq M$. Theorem 10.6(ii) shows that $H \preceq F$. If $G \in \mathcal{G}$ and $F \preceq G$ then $H \preceq G$. Hence $W_G \subseteq V_G \subseteq V_H \subseteq M$ and so $G \in \mathcal{H}$.

Conversely, suppose that \mathcal{H} satisfies condition (10.8). If $F \in \mathcal{H}$ then $G \in \mathcal{H}$ for all G with G in \mathcal{G} and $F \preceq G$: hence $M \supseteq \bigoplus_{G \succ F} W_G = V_F$. Therefore $M = \sum_{F \in \mathcal{H}} V_F$, which is an expectation model. ■

We shall now show that this collection of expectation models satisfies the three principles. Let $M_1 = \bigoplus_{H \in \mathcal{H}_1} W_H$ and $M_2 = \bigoplus_{H \in \mathcal{H}_2} W_H$, where \mathcal{H}_1 and \mathcal{H}_2 both satisfy condition (10.8). Then

$$M_1 + M_2 = \bigoplus_{H \in \mathcal{H}_1 \cup \mathcal{H}_2} W_H.$$

If $F \in \mathcal{H}_1 \cup \mathcal{H}_2$ then $F \in \mathcal{H}_i$ for $i = 1$ or 2 . If $F \preceq G$ then G is also in \mathcal{H}_i and so $G \in \mathcal{H}_1 \cup \mathcal{H}_2$. Hence $\mathcal{H}_1 \cup \mathcal{H}_2$ satisfies condition (10.8) and so the Sum Principle is satisfied.

Because the W -spaces are mutually orthogonal,

$$M_1 \cap M_2 = \bigoplus_{H \in \mathcal{H}_1 \cap \mathcal{H}_2} W_H.$$

If $F \in \mathcal{H}_1 \cap \mathcal{H}_2$ then $F \in \mathcal{H}_1$ and $F \in \mathcal{H}_2$. If $F \preceq G$ also, then $G \in \mathcal{H}_1$ and $G \in \mathcal{H}_2$ and so $G \in \mathcal{H}_1 \cap \mathcal{H}_2$. Hence $\mathcal{H}_1 \cap \mathcal{H}_2$ satisfies condition (10.8) and so the Intersection Principle is satisfied.

Finally, put $\mathcal{H}'_1 = \{H \in \mathcal{H}_1 : H \notin \mathcal{H}_2\}$ and $\mathcal{H}'_2 = \{H \in \mathcal{H}_2 : H \notin \mathcal{H}_1\}$. Then $M_1 \cap (M_1 \cap M_2)^\perp = \bigoplus_{H \in \mathcal{H}'_1} W_H$ and $M_2 \cap (M_1 \cap M_2)^\perp = \bigoplus_{H \in \mathcal{H}'_2} W_H$. Because \mathcal{H}'_1 and \mathcal{H}'_2 have no factors in common, these two spaces are orthogonal to each other, and so the Orthogonality Principle is satisfied.

The diagram showing the relationships between the expectation models is the opposite way up to the Hasse diagram for the factors. Of course, we could draw the Hasse diagram for the factors the other way up too, but then it would no longer correspond to the conventional order in the anova table. In general, the diagram for the expectation models has more points than the Hasse diagram for the factors because a model such as $V_F + V_G$ may not be a factor subspace if neither of F and G is finer than the other.

Example 10.1 revisited (Rye-grass) Here the treatments are all combinations of the factors **cultivar** and **nitrogen**. The Hasse diagram for the treatment factors is in Figure 10.9(e). The expectation models are shown in Figure 10.12(a).

Example 10.5 revisited (Drugs at different stages of development) The treatment structure is shown in Figure 10.10(a). Although this has more treatment factors than the previous example, the diagram of the expectation models, repeated from Figure 3.3 in Figure 10.12(b), looks identical to the previous one.

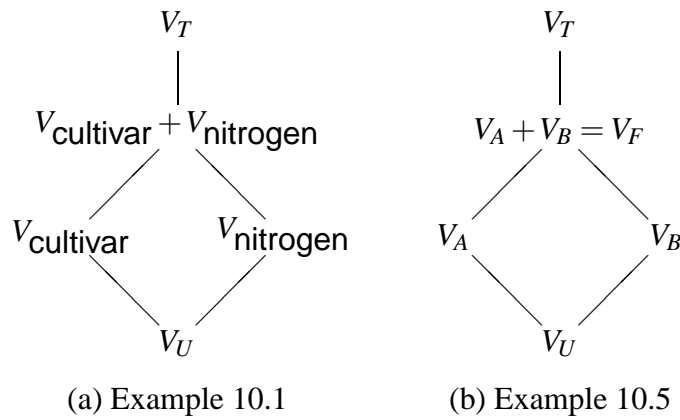


Figure 10.12: Collections of expectation models

Example 10.4 revisited (Nematodes) The Hasse diagram for the treatment factors is in Figure 10.9(g). The expectation models are shown in Figure 10.13(a).

Example 10.6 revisited (Rats) The Hasse diagram for the treatment factors is in Figure 10.10(b). The expectation models are shown in Figure 10.13(b).

If the treatments consist of all levels of three treatment factors then we have the Hasse diagram in Figure 10.11 and the expectation models in Figure 5.10.

Example 10.7 (Main-effects only design in blocks) The design in Section 9.1.3 is for two treatment factors F and G , each with four levels, on the assumption that

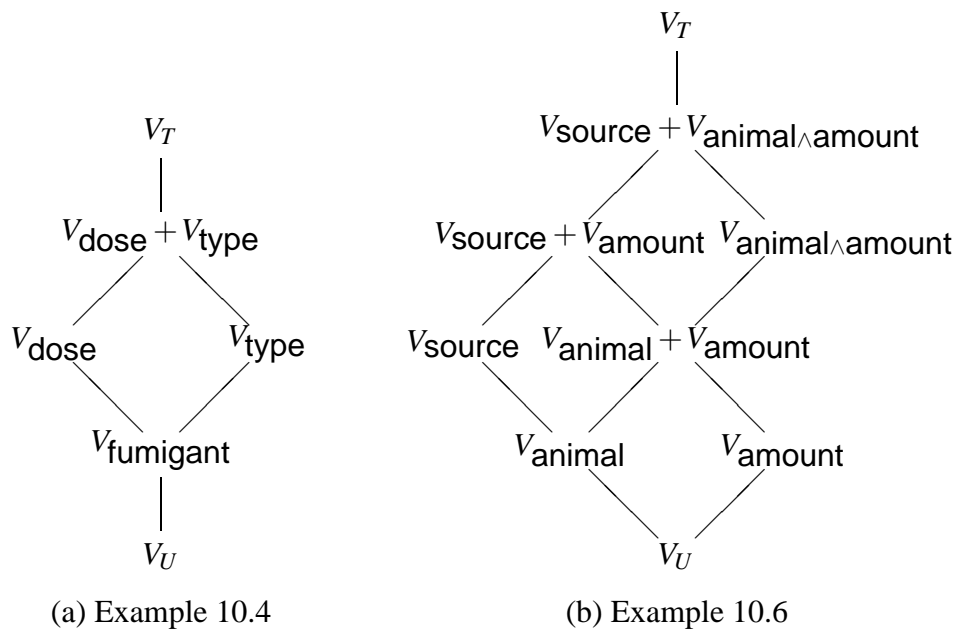


Figure 10.13: More collections of expectation models

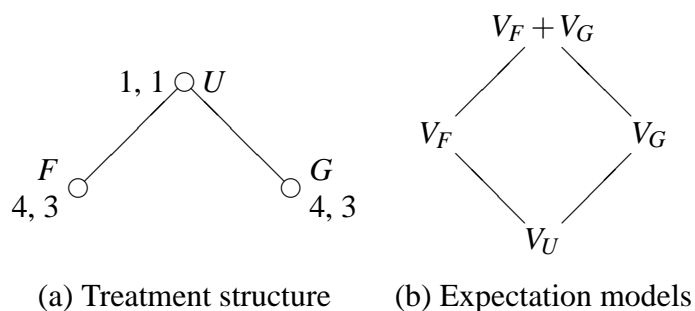


Figure 10.14: Diagrams for Example 10.7

their interaction is zero. Thus the treatment structure is simply the one shown in Figure 10.14(a), with the collection of expectation models in Figure 10.14(b).

10.9 Covariance and strata

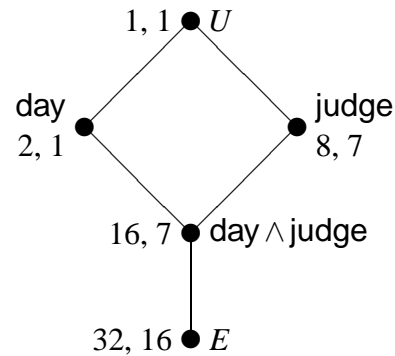
When the plot structure is any more complicated than those in Chapters 2, 4 and 6, it is usual to assume that the plot structure gives random effects rather fixed effects; that is, the plot structure makes no difference to the expectation model but does determine the covariance matrix. Thus we hope for a plot structure which defines a covariance matrix whose strata can be determined.

Since the strata should give an orthogonal decomposition of the whole space including V_0 , a reasonable plot structure must satisfy at least the three conditions for an orthogonal treatment structure. In addition, it must satisfy three more conditions. First, E must be included, so that we obtain a decomposition of the *whole space* V_E . Example 10.7 shows that this condition is not necessary for an orthogonal treatment structure. Secondly, all factors must be uniform. If there are blocks of different sizes then we cannot relabel them by randomization. Furthermore, it may not be reasonable to assume that the covariance between a pair of plots in large block is the same as the covariance between a pair of plots in small block. Again this contrasts with treatment structure, where we have seen several examples of treatment factors that are not uniform. Thirdly, if F and G are both plot factors then so must $F \wedge G$ be. Two further examples should make the reason for this clear.

Example 10.8 (Soap pads) A manufacturer of soap pads tries to improve them by varying the amount of detergent, the solubility of the detergent and the coarseness of the pad. Each of these treatment factors has just two levels, high and low, and so there are eight treatments.

When a quantity of each type of soap pad has been made, the manufacturer compares them in a two-day trial during which each of eight judges tests two soap pads per day and scores them on a subjective scale from 1 to 5. Thus the plot structure is schematically as shown in Figure 10.15(a), while the Hasse diagram for the plot factors is in Figure 10.15(b).

	Day 1	Day 2
Judge 1	○ ○	○ ○
Judge 2	○ ○	○ ○
Judge 3	○ ○	○ ○
Judge 4	○ ○	○ ○
Judge 5	○ ○	○ ○
Judge 6	○ ○	○ ○
Judge 7	○ ○	○ ○
Judge 8	○ ○	○ ○



(a) Layout

(b) Hasse diagram

Figure 10.15: Plot structure in Example 10.8

Consider the pattern of correlations in this plot structure. We might expect a correlation ρ_U between plots which are in different judges and different days, another correlation ρ_J between plots which are in the same judge but different days, and a third correlation ρ_D between plots which are in the same day but different judges. However, the correlation between a pair of plots which are in the same day and the same judge should probably be different from all of the previous three. In other words, we need to take account of the factor $\text{day} \wedge \text{judge}$.

Thinking about randomization leads us to a similar conclusion. Any method of randomization must preserve the grouping of the plots into judges, and it must preserve the grouping of the plots into days. Thus it cannot help but preserve the classes of $\text{day} \wedge \text{judge}$.

Example 10.9 (Cross-over with blocks) Suppose that a cross-over trial for three treatments uses 30 people, of whom half are men and half are women. If sex is considered a relevant plot factor then the correlation between different plots in the same time-period should depend on whether or not they are in the same sex. Then it is logical that the correlation between plots in different time-periods and different subjects should also depend on whether or not they are in the same sex. In other words, the plot factor $\text{period} \wedge \text{sex}$ is relevant.

The Hasse diagram is in Figure 10.16.

Definition A set \mathcal{F} of inequivalent factors on Ω is an *orthogonal plot structure* if

- (i) every factor in \mathcal{F} is uniform;
- (ii) $U \in \mathcal{F}$;
- (iii) $E \in \mathcal{F}$;
- (iv) if $F \in \mathcal{F}$ and $G \in \mathcal{F}$ then $F \vee G \in \mathcal{F}$;

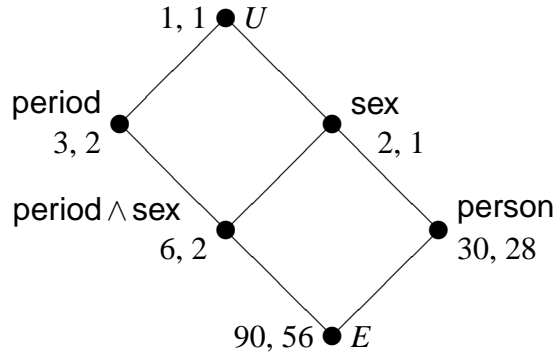


Figure 10.16: Hasse diagram for the plot factors in Example 10.9

- (v) if $F \in \mathcal{F}$ and $G \in \mathcal{F}$ then $F \wedge G \in \mathcal{F}$;
- (vi) if $F \in \mathcal{F}$ and $G \in \mathcal{F}$ then F is orthogonal to G .

For the rest of this section we assume that \mathcal{F} is an orthogonal plot structure.

Suppose that α and β are in Ω . Then $U(\alpha) = U(\beta)$, so there is at least one factor in \mathcal{F} for which α and β are in the same class. If F and G are two such factors then $F \wedge G$ is another. Consequently there is a factor F with this property that is *finest* in the sense that any other factor in \mathcal{F} which has this property must be coarser than F .

As we did in Chapter 4 for the block factor B , and in Chapter 6 for the row factor R and the column factor C , for each factor F in \mathcal{F} we define an $N \times N$ matrix \mathbf{J}_F whose (α, β) -entry is equal to

$$\begin{cases} 1 & \text{if } F(\alpha) = F(\beta) \\ 0 & \text{otherwise.} \end{cases}$$

For an orthogonal plot structure we can also define another $N \times N$ matrix \mathbf{A}_F by putting the (α, β) -entry of \mathbf{A}_F equal to

$$\begin{cases} 1 & \text{if } F \text{ is the finest factor such that } F(\alpha) = F(\beta) \\ 0 & \text{otherwise.} \end{cases}$$

Now, if $F(\alpha) = F(\beta)$ then the finest factor G which has α and β in the same class must satisfy $G \preceq F$. This shows that

$$\mathbf{J}_F = \sum_{G \preceq F} \mathbf{A}_G \quad (10.9)$$

for all F in \mathcal{F} . Thus

$$\mathbf{J}_F = \mathbf{A}_F + \sum_{G \prec F} \mathbf{A}_G,$$

which can be rewritten as

$$\mathbf{A}_F = \mathbf{J}_F - \sum_{G \prec F} \mathbf{A}_G. \quad (10.10)$$

This implies that the \mathbf{A} -matrices can be calculated from the \mathbf{J} -matrices by using the Hasse diagram, but this time starting at the bottom and working up. At the bottom, Equation (10.10) gives

$$\mathbf{A}_E = \mathbf{J}_E = \mathbf{I}.$$

When the \mathbf{A} -matrices for all points below F have been calculated, use Equation (10.10) to calculate \mathbf{A}_F from \mathbf{J}_F . In fact, we do not actually need to do this calculation, but we need to know that it can be done in principle, because it shows that every \mathbf{A} -matrix is a linear combination of the \mathbf{J} -matrices.

Theorem 10.9 *Let \mathcal{F} be an orthogonal plot structure on Ω . Suppose that $\text{Cov}(\mathbf{Y}) = \sum_{F \in \mathcal{F}} \rho_F \sigma^2 \mathbf{A}_F$, for some (unknown) variance σ^2 and (unknown) correlations ρ_F for F in \mathcal{F} (with $\rho_E = 1$). Then the W -subspaces defined in Theorem 10.6 are the eigenspaces of $\text{Cov}(\mathbf{Y})$; that is, they are the strata.*

Proof First we note that, because \mathcal{F} is an orthogonal plot structure it does satisfy the hypotheses of Theorem 10.6. Hence the definition of W -subspaces given there makes sense. Moreover, these spaces are orthogonal to each other and the whole space V is the direct sum of the W_F for F in \mathcal{F} (excluding any W_F that happens to be zero).

Put $\mathbf{C} = \text{Cov}(\mathbf{Y})$. Then $\mathbf{C} = \sum_{F \in \mathcal{F}} \rho_F \sigma^2 \mathbf{A}_F$. Since each \mathbf{A} -matrix is a linear combination of the \mathbf{J} -matrices, there must be constants v_F , for F in \mathcal{F} , such that $\mathbf{C} = \sum_{F \in \mathcal{F}} v_F \mathbf{J}_F$.

Let \mathbf{x} be a vector in V and let F be a factor in \mathcal{F} . An argument like the one in Section 4.6 shows that if $\mathbf{x} \in V_F$ then $\mathbf{J}_F \mathbf{x} = k_F \mathbf{x}$ while if $\mathbf{x} \in V_F^\perp$ then $\mathbf{J}_F \mathbf{x} = \mathbf{0}$. If $\mathbf{x} \in W_F$ then \mathbf{x} is contained in every V_G for which $G \succ F$ but is orthogonal to every other V -space, and hence $\mathbf{C}\mathbf{x} = (\sum_{G \succ F} k_G v_G) \mathbf{x}$. Thus W_F is an eigenspace of \mathbf{C} . ■

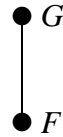
In principle we can calculate the eigenvalues of $\text{Cov}(\mathbf{Y})$ from the covariances $\rho_F \sigma^2$ by using the Hasse diagram twice, but we shall not do so. We simply call the eigenvalues of $\text{Cov}(\mathbf{Y})$ the *stratum variances*, and denote them ξ_F for F in \mathcal{F} . We usually assume that if $F \prec G$ then $\xi_F < \xi_G$.

10.10 Randomization

For almost all orthogonal plot structures in actual use, the Hasse diagram gives the correct method of randomization. Draw the Hasse diagram for all the plot factors (ignoring treatment factors). Randomize by working down the diagram from U to F as follows.

- (i) At U , do nothing, and mark U as “done”.

- (ii) at F , if there is a single line coming down into F like this,



and if G has been marked “done”, then randomize whole classes of F within each class of G ; then mark F as “done”;

- (iii) at F , if there are two or more lines coming down into F and all the points above F have been marked “done”, then do nothing and mark F as “done”;
- (iv) continue until E is “done”.

Although there are some orthogonal plot structures for which this method does not work, the only one that I have come across in practice is the superimposed design like the one in Section 9.3.1. Even this is a problem only when the second design has not been randomized at the same time as the first.

10.11 Orthogonal designs

Each design has three components—the plot structure, the treatment structure and the design function T which allocates treatments to plots. In the last two sections we have examined desirable properties of the first two. Now we can look at the design function.

The first important property of the design function is that treatment factors which are orthogonal to each other when considered as factors on \mathcal{T} should remain orthogonal to each other when considered as factors on Ω . The simplest way to achieve this is with equal replication. If the Hasse diagram for the treatment structure is just a chain, as in Figure 10.6(c), then the treatment structure remains orthogonal on Ω no matter what the replication is. The same is true for treatment structures like that in Example 10.5, where there is one factor splitting the treatments into types and, for each type, another factor distinguishing between the treatments of that type. Any other case of unequal replication needs careful checking for orthogonality. Some possibilities are shown in Figure 10.7.

If G is a treatment factor then we want the treatment subspace W_G to lie in one stratum and hence be orthogonal to the other strata. Thus we need G to be orthogonal to all the plot factors. In particular, T itself must be orthogonal to all the plot factors. In a complicated design, this orthogonality condition needs some checking, but Corollary 10.4 can be used to cut down the work.

In Example 10.1 we saw that $\text{strip} \vee T = \text{cultivar}$. We know from Section 8.3 that the subspace for **cultivar** is estimated in a different stratum from the rest of the treatment space. In general, if F is a plot factor and G is a treatment factor then we

need $F \vee G$ to be a treatment factor so that the subspace for $F \vee G$ can be removed from W_G .

Surprisingly, this third condition is not a real constraint, as we shall now show. If F is a plot factor and G is a treatment factor then $T \preceq G \preceq F \vee G$, so $F \vee G$ does give a grouping of the treatments, and this can simply be added to the list of treatment factors. If $F \vee G$ has no natural meaning on the treatments then some people call it a treatment *pseudofactor* rather than a treatment factor.

Suppose that we simply take all suprema of the form $F \vee G$, where F is a plot factor and G is a treatment factor, and include them in the list of treatment factors. There is no need to repeat the process, because if F_1 and F_2 are plot factors then $F_1 \vee (F_2 \vee G) = (F_1 \vee F_2) \vee G$, which is in the list after the first run, because $F_1 \vee F_2$ is also a plot factor. Moreover, if G_1 and G_2 are treatment factors then so is $G_1 \vee G_2$: now $(F_1 \vee G_1) \vee (F_2 \vee G_2) = (F_1 \vee F_2) \vee (G_1 \vee G_2)$, which is also in the new list of treatment factors. So long as every plot factor is orthogonal to every (original) treatment factor, orthogonality is not violated by the inclusion of these new treatment factors: Corollary 10.4 shows first that $F_1 \vee G_1$ is orthogonal to F_2 , second that $F_1 \vee G_1$ is orthogonal to G_2 , third that $F_1 \vee G_1$ is orthogonal to $F_2 \vee G_2$.

Definition A design whose plot structure consists of a set \mathcal{F} of factors on a set Ω of plots, whose treatment structure consists of a set \mathcal{G} of factors on a set \mathcal{T} of treatments and whose treatments are allocated to plots according to a design function T is an *orthogonal design* if

- (i) \mathcal{F} is an orthogonal plot structure;
- (ii) \mathcal{G} is an orthogonal treatment structure;
- (iii) the function T is such that
 - (a) if treatment factors G_1 and G_2 in \mathcal{G} are orthogonal to each other on \mathcal{T} then they remain orthogonal to each other on Ω ;
 - (b) if $F \in \mathcal{F}$ and $G \in \mathcal{G}$ then F is orthogonal to G ;
 - (c) if $F \in \mathcal{F}$ and $G \in \mathcal{G}$ then $F \vee G \in \mathcal{G}$.

In principle there are twelve conditions to check to verify that a design is orthogonal. Fortunately, good statistical computing packages can do this for you. I recommend always doing such a computer check before giving out a plan for a complicated experiment.

All the designs that we have met so far are orthogonal.

An orthogonal plot structure determines the null analysis of variance. There is one line for each stratum, showing the name F and the number of degrees of freedom d_F . Conventionally, the line for F is written lower than the line for G if $F \prec G$. In particular, E is always written at the bottom, which explains why W_E is sometimes called the *bottom stratum*.

The null analysis-of-variance table is expanded to calculations ignoring treatments by writing in the sum of squares, $SS(F)$, for each F in \mathcal{F} . These sums of squares are calculated from the crude sums of squares by using the Hasse diagram for \mathcal{F} .

To obtain the skeleton analysis of variance from the null analysis of variance, we have to decide which stratum to put each treatment subspace in. This is straightforward if all treatment subspaces are estimated in the same stratum, as in a complete-block design or the second version of the calf-feeding trial. In other cases we do not want to have to go through an argument like the proof of Theorem 8.2. Fortunately, the next result shows that the Hasse diagrams can be used to locate the treatment subspaces in the appropriate strata.

Theorem 10.10 *Let G be a treatment factor in an orthogonal design. Then there is a unique plot factor F finer than or equivalent to G which is coarsest in the sense that every other plot factor which is finer than G is also finer than F . Moreover, $W_G \subseteq W_F$.*

Proof The equality factor E is in \mathcal{F} and $E \preceq G$ so there is at least one plot factor finer than or equivalent to G . If F_1 and F_2 are two such plot factors then $F_1 \vee F_2$ is also a plot factor and $F_1 \vee F_2 \preceq G$. Consequently there is a coarsest plot factor F finer than or equivalent to G .

Since $F \preceq G$, we have $W_G \subseteq V_G \subseteq V_F$. Suppose that H is a plot factor with $F \prec H$. Then we cannot have $H \preceq G$, and so $H \vee G \neq G$. Put $P = H \vee G$. Then $G \prec P$. Since the design is orthogonal, P is a treatment factor and so $W_G \subseteq V_G \cap V_P^\perp = V_G \cap V_{H \vee G}^\perp$, which is orthogonal to V_H because the factors G and H are orthogonal to each other. Hence

$$W_G \subseteq V_F \cap \bigcap_{H \succ F, H \in \mathcal{F}} V_H^\perp = W_F. \quad \blacksquare$$

Thus we locate treatment subspaces by combining the two Hasse diagrams into one, using a composite type of dot to indicate any factor that is in both \mathcal{F} and \mathcal{G} . For each treatment factor, simply find the coarsest plot factor which is below it or equivalent to it.

Again, good statistical computing packages can calculate the skeleton analysis of variance before there are any data.

When the plots are unstructured the combined Hasse diagram is very simple. Figure 10.17 shows the three diagrams for Example 5.10.

At first sight, the complete-block design in Example 4.5, the first version of the calf-feeding trial in Example 8.1 and the improperly replicated experiment in Example 10.3 are quite similar. All have plot structure like the one in Figure 10.9(a) and treatment structure like the one in Figure 10.8(b). However, the combined Hasse diagrams, which are shown in Figure 10.18, immediately show the differences between these three cases. In Example 4.5 all treatment differences are estimated in the bottom stratum. In Example 8.1 all treatment differences are estimated in the

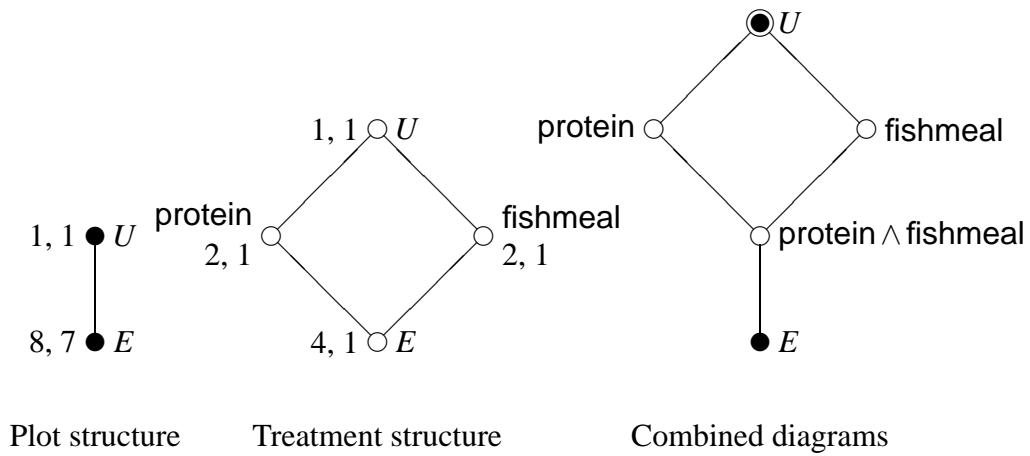


Figure 10.17: Three Hasse diagrams for Example 5.10

middle (pen) stratum. In Example 10.3 they are also estimated in the middle stratum, but now treatments are aliased with a plot factor and so there is no residual mean square for testing significance or for estimating variance.

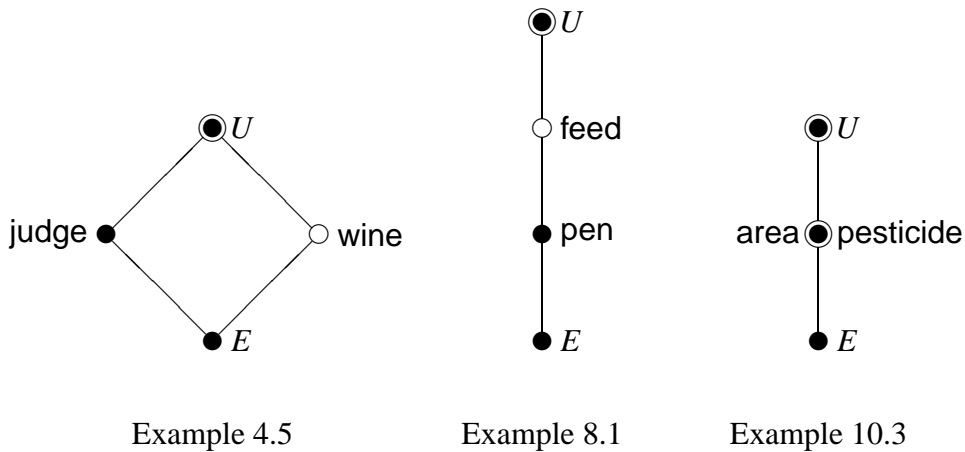
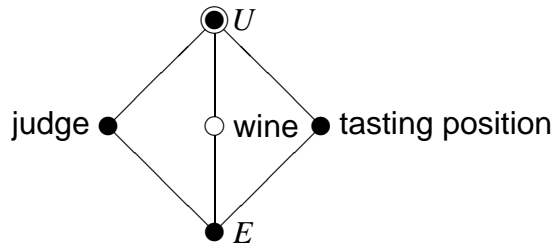


Figure 10.18: Combined Hasse diagrams for three superficially similar experiments

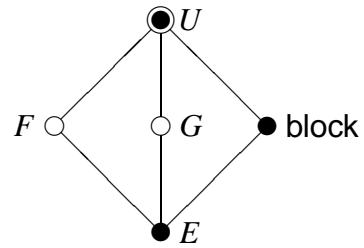
Combined diagrams for four more complicated examples are shown in Figure 10.19. It can be verified that each combined Hasse diagram leads to the skeleton analysis of variance given previously.

Example 10.1 revisited (Rye-grass) We have seen that $\text{strip} \vee T = \text{cultivar}$. Each treatment occurs in each field so $\text{field} \vee T = U$. Therefore $\text{field} \vee \text{cultivar}$ and $\text{field} \vee \text{nitrogen}$ are also equal to U . Every level of nitrogen occurs on each strip, so $\text{strip} \vee \text{nitrogen} = U$. Hence we obtain the combined Hasse diagram in Figure 10.20. This is consistent with the skeleton anova in Table 8.13.

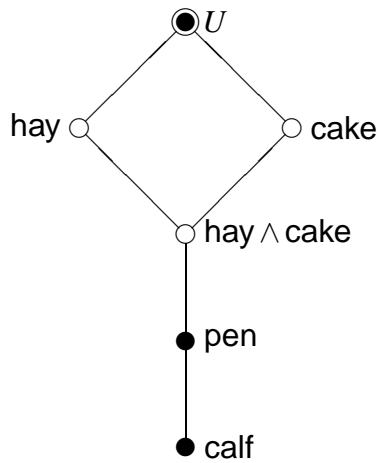
To turn the skeleton analysis of variance into the full analysis of variance, we need to calculate expected mean squares (for the theoretical version) and treatment



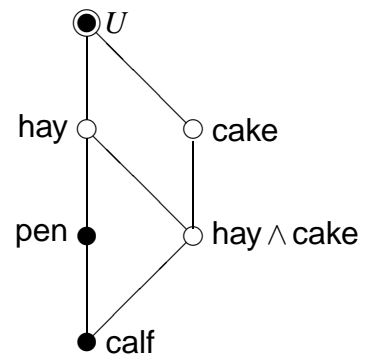
(a) Example 6.1
See Table 6.4



(b) Example 10.7
See Table 9.3



Example 8.1, second version
See Table 8.5



Example 8.1, third version
See Table 8.8

Figure 10.19: Combined Hasse diagrams with reference to the relevant skeleton anovas

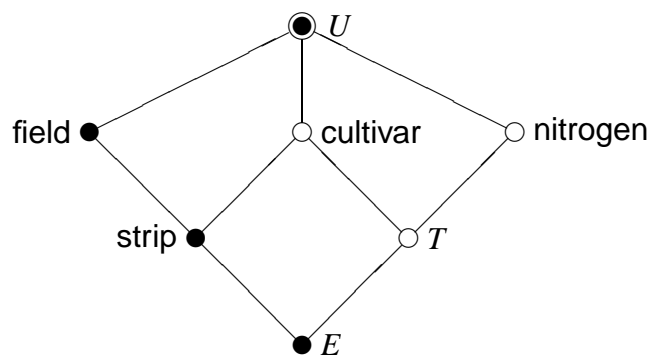


Figure 10.20: Combined Hasse diagram in Example 10.1

sums of squares and residual sums of squares (for arithmetic on the data). Write τ for $\mathbb{E}(\mathbf{Y})$ and τ_G for $P_{W_G}\tau$. Theorem 2.10(ii) shows that if treatment subspace W_G is contained in stratum W_F then

$$\mathbb{E}(\|P_{W_G}\mathbf{Y}\|^2) = \|\tau_G\|^2 + d_G\xi_F.$$

If W_G is the only treatment subspace in stratum W_F then, in the full analysis of variance, the line for stratum F is as follows.

Stratum	source	df	SS	EMS	VR
F	G	d_G	$SS(G)$	$\frac{\ \tau_G\ ^2}{d_G} + \xi_F$	$\frac{MS(G)}{MS(\text{residual in } F)}$
	residual	by subtraction		ξ_F	—
	total	d_F	$SS(F)$		

Here $SS(G)$ is calculated from crude sums of squares by using the Hasse diagram for treatment factors, and the residual degrees of freedom and sum of squares are calculated by subtraction. The appropriate variance ratio to test the effect of G is

$$\frac{MS(G)}{MS(\text{residual in } F)}.$$

The estimator of ξ_F is $MS(\text{residual in } F)$: this estimate is used in the formula for standard errors of contrasts in W_G .

If there is more than one treatment subspace in stratum W_F then they are all written in the full analysis-of-variance table before the residual sum of squares for that stratum is calculated.

It is important to remember that, just as for degrees of freedom, effects and sums of squares, the *meaning* of the source labelled by factor G in the full analysis-of-variance table depends on what other factors are included. The line for G is there to help us answer the question ‘do the different levels of treatment factor G cause differences in response over and above those caused by treatment factors H coarser than G ?’

In particular, do not confuse the *factor* $F \wedge G$ with the F -by- G interaction. It is true that in many experiments the effect of $F \wedge G$ is indeed the F -by- G interaction. However, it is not unless both F and G are included in \mathcal{G} . Even when F and G are both in \mathcal{G} , the effect of $F \wedge G$ is not the whole of the F -by- G interaction if there is another treatment factor coarser than $F \wedge G$. An example of this will be given at the end of the next section.

10.12 Further examples

This section contains a collection of new examples of orthogonal designs. It is not exhaustive, but does demonstrate the versatility of the methods developed in

this chapter.

Example 10.10 (Example 1.8 continued: Mental arithmetic) The treatments are all combinations of group size (whole class or four children) and timing (one hour per week or 12 minutes per day). The Hasse diagram for the treatment factors is in Figure 10.21(b).

Suppose that five schools take part in the experiment, each of these schools having four classes of children of the appropriate age. If there are 20 children per class, we obtain the Hasse diagram for plot factors in Figure 10.21(a).

Teaching methods can only be applied at the whole-class level, but we do not want differences between schools to bias our conclusions. Therefore, the four treatments are randomized to the four classes independently within each school. Then $\text{class} \prec T$ but T is orthogonal to school and $T \vee \text{school} = U$. Thus we obtain the skeleton analysis of variance in Table 10.4.

The one unrealistic assumption here is that all the classes have exactly 20 children. A possible way forward if the classes have different numbers of children is to use the average test mark per class as the response, since treatments are in any case estimated in the class stratum. However, these averages will not have the same variance if they are taken over different numbers of children, so this method should not be used unless the class sizes are reasonably similar.

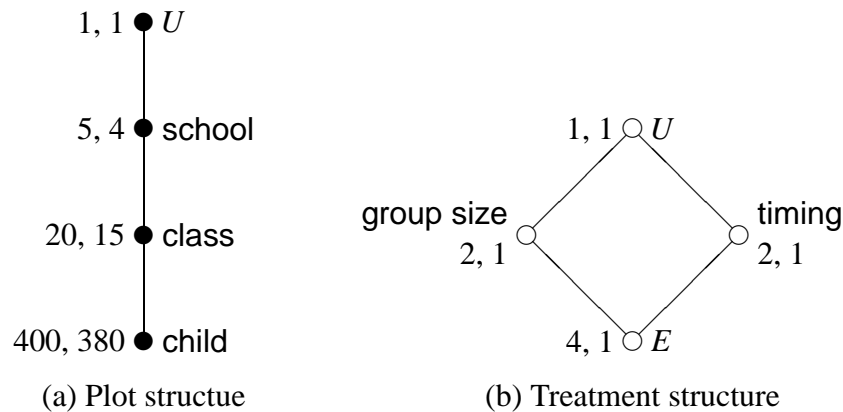


Figure 10.21: Hasse diagrams for Example 10.10

Example 10.11 (Bean weevils) An experiment was conducted to investigate the effects of certain behaviour-modifying chemicals on bean weevils. One chemical was a pheromone intended to attract the insects. This was released at either $5\mu\text{g}$ per day or $240\mu\text{g}$ per day. The other chemical was neem oil, which is intended to deter the insects from feeding and so make them move elsewhere. This was applied at 4kg/ha , dissolved in water at the rate of either 10 or 25 litres per hectare. For comparison there was an untreated control.

The treatments are summarized in Table 10.5. Since some treatments are designed to *increase* the number of weevils relative to the control, while others are

Stratum	source	degrees of freedom
mean	mean	1
school	school	4
class	group size	1
	timing	1
	group size \wedge timing	1
	residual	12
	total	15
child	child	380
Total		400

Table 10.4: Skeleton analysis of variance for Example 10.10

designed to *decrease* it, it is not appropriate to have a treatment factor which distinguishes between the control and the rest. However, it is appropriate to have a treatment factor **type** which distinguishes between the three types of treatment, and two further factors, **pheromone** and **neem**, which distinguish between treatments of a single type. These are shown in Table 10.5 and Figure 10.22(b).

treatment	untreated	pheromone $5\mu\text{g}$ per day	pheromone $240\mu\text{g}$ per day	neem oil in 10 litres	neem oil in 25 litres
type	1	2	2	3	3
pheromone	1	2	3	4	4
neem	1	2	2	3	4

Table 10.5: Treatments in Example 10.11

Beans were sown in a field in the spring. It was impossible to predict the direction from which the insects would arrive, so a row-column design was used. The plot structure is in Figure 10.22(a). The experimenters wanted double the replication for the control treatment, and so the plots were arranged in a 6×6 square. The design chosen was like a Latin square, except that the control treatment occurred twice in each row and twice in each column. In fact, the design was constructed and randomized using a 6×6 Latin square in which two of the letters had been assigned to the control treatment.

Theorem 4.1 shows that all treatment factors are orthogonal to rows and columns and that $T \vee \text{row} = T \vee \text{column} = U$. Hence we obtain the skeleton analysis of variance in Table 10.6.

This experiment has a further complication. If one treatment is successful in attracting insects then the weevils may spread to neighbouring plots. To avoid bias, we should have the successful treatment next to each other treatment equally often. There are ways of achieving such *neighbour balance*, but they are beyond the scope of this book.

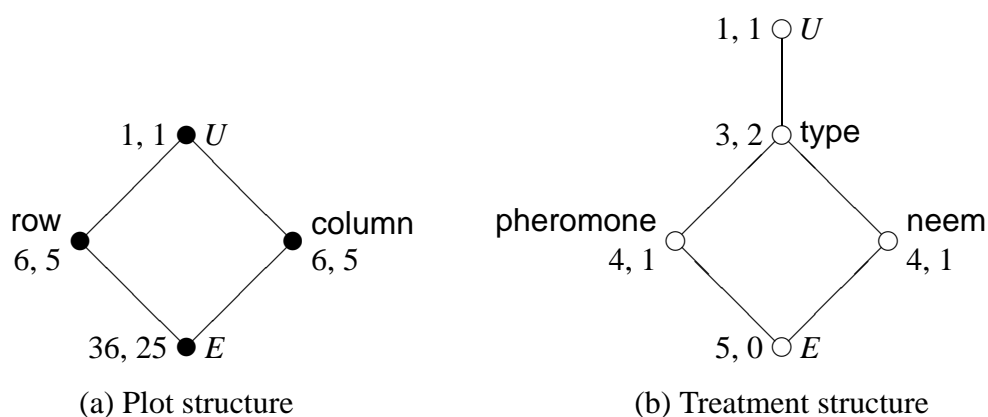


Figure 10.22: Hasse diagrams for Example 10.11

Stratum	source	degrees of freedom
mean	mean	1
row	row	5
column	column	5
plot	type	2
	pheromone	1
	neem	1
	residual	21
	total	25
Total		36

Table 10.6: Skeleton analysis of variance for Example 10.11

Example 10.12 (Rugby) A rugby coach wants to test whether rugby players run faster with the ball in their left hand, in their right hand, or held in both hands. He selects twenty right-handed rugby players and gives them two sessions of test runs. In each session, each player makes four runs of 50 metres, being timed from the 10-metre line to the 40-metre line. In one run he carries the ball in his left hand; in one run in his right; in one run in both. For comparison, he makes one run with no ball. In each session, the treatments are allocated to the four runs by twenty players according to a row-column design made of five Latin squares of order 4.

The plot factors are U , **player**, **session**, **run**, **player** \wedge **session**, and **player** \wedge **run**, which is E . These are shown on the Hasse diagram in Figure 10.23(a), which gives the null analysis of variance in Table 10.7.

The set of four treatments is best regarded not as a 2×2 factorial but as being divided by the factor **number** according to the number of hands used. This gives the Hasse diagram in Figure 10.23(b). The single degree of freedom for T is for the comparison between right and left hands.

The Latin square construction in each session ensures that each treatment occurs once in each class of **player** \wedge **session** and five times in each run. Thus treatments also occur equally often in each session and in each player, and so the treatment

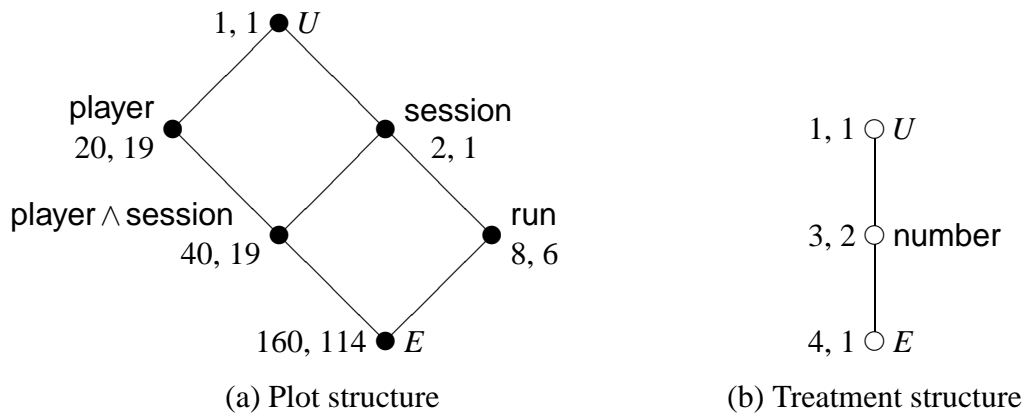


Figure 10.23: Hasse diagrams for Example 10.12

Stratum	df
mean	1
player	19
session	1
player \wedge session	19
run	6
player \wedge run	114
Total	160

Table 10.7: Null analysis of variance in Example 10.12

space lies in the bottom stratum W_E . Thus we obtain the skeleton analysis of variance in Table 10.8.

Example 10.13 (Carbon dating) An archaeology organization wishes to examine the consistency of three different methods of carbon dating: liquid scintillation counting, gas proportional counting and accelerator mass spectrometry. Thirty laboratories are willing to take part in the experiment. The organization has available 11 sets of equipment for liquid scintillation counting, 6 for gas proportional counting and 13 for accelerator mass spectrometry. These sets of equipment are allocated at random to the 30 laboratories, and technicians at each are trained in their use.

The organization has eight test items for dating, taken from different archaeological sites and composed of different substances. Each laboratory will use its equipment to carbon-date each of the test items.

The plot factors are U , laboratory and test, which is E . The Hasse diagram is in Figure 10.24(a). This gives the null analysis of variance in Table 10.9.

The treatment factors are U , method, item and method \wedge item, which is T . The organization is chiefly interested in the consistency of the methods in dating the different samples, that is, in the method-by-item interaction. The Hasse diagram is in Figure 10.24(b).

Stratum	source	degrees of freedom
mean	mean	1
player	player	19
session	session	1
player \wedge session	player \wedge session	19
run	run	6
player \wedge run	number	2
	T	1
	residual	111
	total	114
Total		160

Table 10.8: Skeleton analysis of variance for Example 10.12

Treatments are not equi-replicated. They have the following replications.

method	item							
	1	2	3	4	5	6	7	8
liquid scintillation counting	11	11	11	11	11	11	11	11
gas proportional counting	6	6	6	6	6	6	6	6
accelerator mass spectrometry	13	13	13	13	13	13	13	13

Theorem 10.5 shows that **method** is orthogonal to **item** on the set of plots. Within each method, every possible combination of T and **laboratory** occurs just once, so T is orthogonal to **laboratory** and $T \vee \text{laboratory} = \text{method}$. Each item is tested once by each laboratory and so **item** is orthogonal to **laboratory** and $\text{item} \vee \text{laboratory} = U$. Thus we obtain the skeleton analysis of variance in Table 10.10.

Apart from the unequal replication, this example is the same as the third version of Example 8.1.

Example 10.14 (Example 4.1 continued: Irrigated rice) Figure 4.1 shows 32 plots in a rice paddy in an 8×4 rectangular array. The columns correspond to irrigation channels, so should be used as a system of blocks. The rows correspond to distance

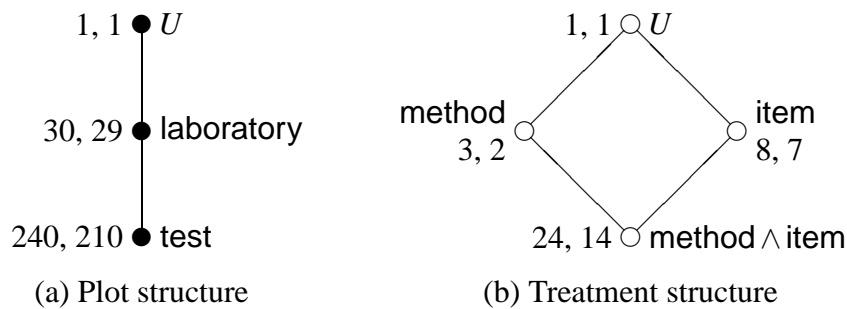


Figure 10.24: Hasse diagrams for Example 10.13

A	B	C	D
E	F	G	H
D	A	B	C
H	E	F	G
C	D	A	B
G	H	E	F
B	C	D	A
F	G	H	E

Figure 10.25: An orthogonal design for eight treatments in Example 10.14

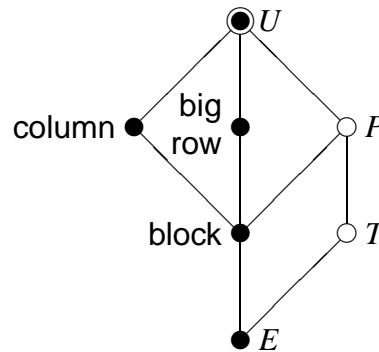


Figure 10.26: Combined Hasse diagram for Example 10.14

from the main irrigation channel, but this is a continuous source of variation so we have some freedom over how many rows to put together to give blocking in this direction. If there are eight treatments then it is convenient to put the rows into pairs, so that each treatment can occur once in each column and once in each pair of rows. A design with this property is called a *semi-Latin square*.

The Hasse diagram for the plot factors is similar to the one in Figure 10.15(b). Let us call a pair of rows a *big row* and the 16 classes of $\text{big row} \wedge \text{column}$ *blocks*. Theorem 10.5 shows that the only way that we can have the treatment factor orthogonal to blocks is for any two blocks to either contain the same two treatments or have no treatments in common. In other words, we must have a design like the one in Figure 10.25, which is made from a Latin square of order 4 by replacing each letter by two letters.

This design should be randomized by randomizing big rows, randomizing columns and then randomizing the pair of plots within each block.

To analyse the design, we need the factor P which groups the eight treatments into the four pairs $\{A, E\}$, $\{B, F\}$, $\{C, G\}$ and $\{D, H\}$. Then $P = \text{block} \vee T$ and $\text{block} \prec P$. However, $\text{big row} \vee T = \text{column} \vee T = U$, so we obtain the combined Hasse diagram in Figure 10.26 and the skeleton analysis of variance in Table 10.11.

The three degrees of freedom for P are estimated less precisely than the remaining treatment degrees of freedom, so this design may not be a good choice. In fact, there are better non-orthogonal designs, but they will not be covered in this book.

Stratum	df
mean	1
laboratory	29
test	210
Total	240

Table 10.9: Null analysis of variance in Example 10.13

Chapter 10. The Calculus of Factors

Stratum	source	degrees of freedom
mean	mean	1
laboratory	method	2
	residual	27
	total	29
test	item	7
	method \wedge item	14
	residual	189
	total	210
Total		240

Table 10.10: Skeleton analysis of variance for Example 10.13

Stratum	source	degrees of freedom
mean	mean	1
column	column	3
big row	big row	3
block	P	3
	residual	6
	total	9
plot	T	4
	residual	12
	total	16
Total		32

Table 10.11: Skeleton analysis of variance for Example 10.14

Example 10.7 revisited (Main-effects only design in blocks) The original version of this design was a single replicate in four blocks of four plots each. We had to assume that the F -by- G interaction was zero. However, if we double up the design by repeating the original four blocks then we can also estimate the interaction. Note that all eight blocks should be randomized amongst themselves.

Let Q be the factor that groups the sixteen treatments into the four sets of four that were allocated to blocks in the original design. In the new design, $\text{block} \vee T = Q$. Now, $Q \vee F = Q \vee G = U$ and Q is orthogonal to both F and G . Thus if the treatment factors are U, F, G, Q and T then W_Q is part of the F -by- G interaction. This gives the combined Hasse diagram in Figure 10.27 and the skeleton analysis of variance in Table 10.12.

Chapter 12 expands on this idea of splitting up interactions.

Questions for Discussion

10.1 Consider the data set in Question 4.1. Use the factors fumigant, dose and type to decompose the treatment sum of squares into four parts, and calculate the

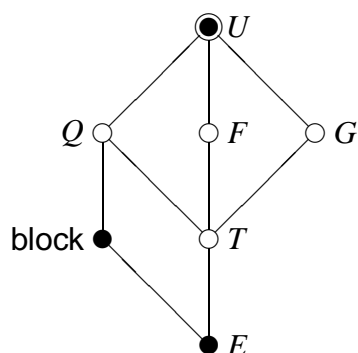


Figure 10.27: Combined Hasse diagram for Example 10.7 revisited

Stratum	source	degrees of freedom
mean	mean	1
block	Q (part of the F -by- G interaction)	3
	residual	4
	total	7
plot	F	3
	G	3
	$F \wedge G$ (rest of the F -by- G interaction)	6
	residual	12
	total	24
Total		32

Table 10.12: Skeleton analysis of variance for Example 10.7 revisited

corresponding tables of means. Use this information with that calculated in Question 4.1 to interpret the complete analysis of variance.

10.2 In each of the following examples, draw the Hasse diagram for the treatment factors, showing numbers of levels and degrees of freedom.

- (a) Example 1.14.
- (b) Example 1.15.
- (c) Example 3.2.
- (d) Example 3.4.
- (e) Example 5.2.
- (f) Example 5.8.
- (g) Example 10.8.

10.3 For Example 8.5,

- (a) Draw the Hasse diagram for the factors on the plots (ignoring treatment factors).
- (b) Draw the Hasse diagram for the factors on the treatments.
- (c) Hence write down the skeleton analysis-of-variance table.

10.4 In the experiment discussed in Section 8.2, four feed treatments were tested on 80 calves. The calves were kept in 8 pens, 10 animals per pen. The feeds were all combinations of two types of hay with two types of cake. Each type of hay was put into four pens for the calves to eat ad lib. Each type of cake was given to 5 calves in each pen.

- (a) Draw the Hasse diagram for the factors on the plots (ignoring treatment factors).
- (b) Draw the Hasse diagram for the factors on the treatments.
- (c) Write down the skeleton analysis-of-variance table.

10.5 An experiment was carried out to find out if so-called ‘non-herbicidal’ pesticides affect photosynthesis in plants. Six pesticides were compared: diuron, carbofuran, tributyltin chloride, chlorpyrifos, phorate and fonofos. Each of these was dissolved in water at five different concentrations. In addition, plain water was used as a control treatment.

Two petri dishes were used for each treatment. Each petri dish was filled with pesticide solution or water. Five freshly cut leaves from mung beans were floated on the surface of the solution in each dish. After two hours, the chlorophyll fluorescence of each leaf was measured.

- (a) Draw the Hasse diagram for the factors on the plots (ignoring treatment factors).
- (b) Draw the Hasse diagram for the factors on the treatments.
- (c) Write down the skeleton analysis-of-variance table.

10.6 Consider the word-processing experiment in Question 6.2. Now suppose that only one copy of each word-processor is available, so that the experiment must last three weeks. The typists will be split into three groups according to experience; each week one group will try out the word-processors.

Furthermore, we now know that the word-processors are of two kinds. Three are WYSIWYG and two are SGML.

- (a) Draw the Hasse diagram for the factors on the plots (ignoring treatment factors).
- (b) Draw the Hasse diagram for the factors on the treatments.

- (c) Describe how to construct the design for the experiment.
- (d) Describe how to randomize the design.
- (e) Construct the skeleton analysis-of-variance table.

10.7 In an experiment into the digestibility of stubble, four feed treatments are to be applied to sheep. There are 16 sheep, in four rooms of four animals each. There are four test periods of four weeks each, separated by two-week recovery periods. Each sheep is to be fed all treatments, one in each test period. During the recovery periods all animals will receive their usual feed, so that they will return to normal conditions before being subjected to a new treatment.

Draw the Hasse diagram for the non-treatment factors involved. Describe how you would construct a suitable design and randomize it. Write down the skeleton analysis-of-variance table, showing stratum, source and degrees of freedom.

10.8 Consider Example 10.8. Suppose that the quantity of detergent is not altered, so that there are only four treatments.

- (a) Show how to allocate treatments to plots in such a way that every judge assesses all four treatments, all treatments are assessed by the same number of judges per day, and each judge in each day sees both levels of solubility and both levels of coarseness.
- (b) Draw the combined Hasse diagram for the plot and treatment factors. Hence write down the skeleton analysis-of-variance table, showing stratum, source and degrees of freedom.
- (c) Describe how you would randomize the design.