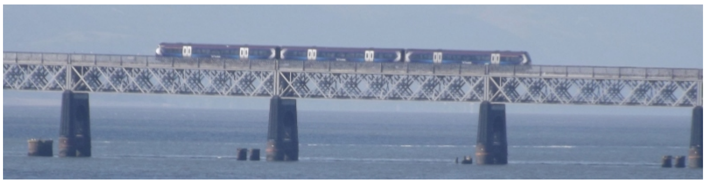
Synchronization, graph homomorphisms, and combinatorics

Peter J. Cameron



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Automata and synchronization

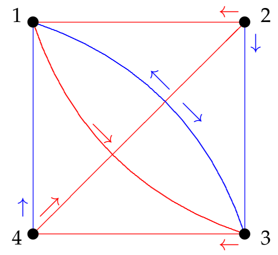
An automaton is

- Computationally, a black box with a number of internal states, which undergoes transitions between states when commands are read;
- Combinatorially, an edge-coloured directed graph with a unique edge of each colour leaving each vertex;
- Algebraically, a semigroup of transformations on the set of states with a distinguished generating set.

An automaton is synchronizing if there is a sequence of transitions which brings it into the same state no matter where it started; equivalently, as a semigroup, it contains a transformation of rank 1 (a constant map).

The cave

You are in a dungeon consisting of a number of rooms. Passages are marked with coloured arrows. Each room contains a special door; in one room, the door leads to freedom, but in all the others, to instant death. You have a schematic map of the dungeon, but you do not know where you are.



You can check that (Blue, Red, Blue, Blue) takes you to room 3 no matter where you start.

Graph homomorphisms

A homomorphism from a graph X to a graph Y is a map from the vertex set of X to the vertex set of Y which maps edges to edges.

The clique number $\omega(X)$ of a graph is the size of the largest complete subgraph of X, and the chromatic number of X is the smallest number of colours in a proper colouring.

- There is a homomorphism $K_r \to X$ if and only if $\omega(X) \geq r$;
- There is a homomorphism $X \to K_r$ if and only if $\chi(X) \leq r$.

An endomorphism of a graph X is a homomorphism from X to X. The endomorphisms of a graph X form a semigroup $\mathrm{End}(X)$.

The obstruction to synchronization

Theorem

A transformation semigroup S is non-synchronizing if and only if there is a non-null graph X, with vertex set the set of states of S, such that

- $S \leq \mathrm{End}(X)$;
- $\omega(X) = \chi(X)$.

A graph X with $\omega(X) = \chi(X) = r$ has endomorphisms onto its r-cliques (these are r-colourings using the vertices of the clique as colours).

Permutation groups

We are especially interested in automata where all but one of the transitions are permutations. The permutations generate a permutation group G.

A permutation group is called synchronizing if the semigroup $\langle G, f\rangle$ is synchronizing for *any* non-permutation f.

Corollary

A permutation group G is non-synchronizing if and only if it is contained in the automorphism group of a non-null graph X with $\omega(X) = \chi(X)$. Any map not synchronized by G is an endomorphism of some such graph.

Latin squares

Let G be the automorphism group of the $n \times n$ square grid X. The only two G invariant graphs are X and its complement $\overline{X}$.

- $\omega(X) = \chi(X) = n$. The n-cliques are the rows and columns of the grid; the n-colourings are the $n \times n$ Latin squares. So each Latin square gives an endomorphism of X not synchronized by G, unique up to choice of image and ordering of colours.
- $\omega(\overline{X}) = \chi(\overline{X}) = n$. The n-cliques correspond to permutations of $1, \ldots, n$, and there are just two n-colourings, the rows and columns of the grid.

Note that X has a huge endomorphism semigroup, whose non-invertible elements correspond to $n \times n$ Latin squares.

A semigroup of Latin squares?

So we have given algebraic structure to the set of Latin squares. Unfortunately it isn't very interesting ...
Let X be the square grid, $G = \mathrm{Aut}(X)$ and $S = \mathrm{End}(X)$. For $f \in S \setminus G$, $\langle G, f \setminus G \rangle$ is, essentially, an equivalence class of Latin squares under row and column permutations and transposition. If we adjoin k elements $f_1, \ldots, f_k$, then the Latin square corresponding to any word in $f_1, \ldots, f_k$ and elements of G is just the one corresponding to the first f_i in the word.
So we have the trivial semigroup, where the product of any two elements is the first one, "blown up" a bit.
Note that any set of equivalence classes of Latin squares defines a subsemigroup. So the number of subsemigroups of S is exponential in $|S|$. This can't happen for groups, where a group of order n has at most $n^{\log n}$ subgroups.

1-factorizations of K_n

Let G be the automorphism group of the line graph of K_n.
There are two G-invariant graphs: the line graph X, and its complement $\overline{X}$.
If n is even, then $\omega(X) = \chi(X) = n - 1$; the colourings correspond to 1-factorizations of K_n (of which, again, there are many; so this graph also has a huge endomorphism semigroup.
The graph $\overline{X}$ never has clique and chromatic number equal if $n \geq 4$. Thus G is synchronizing if and only if n is odd.

1-factorizations of complete uniform hypergraphs

Baranyai's Theorem asserts that the complete r-uniform hypergraph on n vertices has a 1-factorization if and only if r divides n. So S_n acting on r-sets is non-synchronizing if $r \mid n$.
As far as I know there are no good estimates for how many 1-factorizations this hypergraph has.

Large sets of Steiner triple systems

If $v \equiv 1$ or $3 \pmod 6$ and $v > 7$, there is a large set of Steiner triple systems of order v (a partition of the set of all 3-sets into Steiner triple systems). These give rise to colourings of the graph whose vertices are the triples, joined if they meet in two points. So, again, this graph has many endomorphisms. (And again, there are no good estimates of how many, as far as I know.)
It can be shown that, for $v > 8$, these two situations are the only cases where S_n acting on 3-sets fails to be synchronizing. So this group is synchronizing if and only if $v \equiv 2, 4, 5 \pmod 6$ and $v > 8$.

Polar spaces

This example is for the finite geometers.
A classical group, acting on the points of its polar space, has two invariant graphs: the orthogonality graph X (relative to the defining sesquilinear or quadratic form) and its complement $\overline{X}$.

- X has clique number equal to chromatic number if and only if the polar space has a partition into ovoids.
- $\overline{X}$ has clique number equal to chromatic number if and only if the polar space has both an ovoid and a spread.

Despite a lot of work in the last 40 years, we don't know for which polar spaces either of the above conditions hold! The usual remarks about endomorphisms apply here.