Homogeneous Cayley objects

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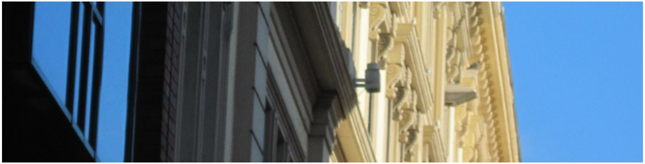

Cayley graphs

A Cayley graph for a group G is a graph Γ on G which is invariant under the action of G by right multiplication. Equivalently, it is a graph which admits G acting regularly on its vertex set as a group of automorphisms.
Note that, unlike in geometric group theory, I do *not* assume that the graph is connected or locally finite.
A Cayley graph is normal if it also admits the action of G by left multiplication. Note that every Cayley graph for an abelian group is normal.

Cayley objects

There is nothing special about graphs in this definition.

Accordingly, if S is a structure of any kind (a relational structure like an order, a first-order structure like a graph, or even more general), whose underlying set is the group G, we say that S is a Cayley object for G if it is invariant under the action of G by right multiplication.

Equivalently, S is a Cayley object for G if it admits the regular action of G as automorphisms.

I will be concerned here only with relational structures.

We could define a normal Cayley object, but I will require this only for graphs.

Why are Cayley objects interesting?

- If a structure is a Cayley object for a group, then we have a group structure on the object concerned. This is interesting for structures like the Urysohn metric space.
- We may be able to use the group to give an explicit description of an object constructed in some indirect way.
- Cayley objects are relevant to various problems about infinite groups, such as the existence of countable B-groups.

Homogeneous structures

A structure S on a set X is homogeneous if every isomorphism between finite (induced) substructures of S can be extended to an automorphism of S.

Homogeneous structures arise in many areas including logic (model-completeness), Ramsey theory, and topological dynamics (extremely amenable groups).

Example

$(\mathbb{Q}, <)$, the set of rational numbers with its usual order, is homogeneous. For any isomorphism between finite sets has the form $a_i \mapsto b_i$ for $i = 1, \ldots, n$, where $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$, and this can be extended to a piecewise-linear order-preserving map on $\mathbb{Q}$.

I will be concerned only with relational structures here.

Fraïssé's Theorem

The age of a structure S is the class of all finite structures which can be embedded in S. Now Fraïssé's Theorem states:

Theorem

A class $\mathcal{C}$ is the age of a countable homogeneous structure S if and only if $\mathcal{C}$ is closed under isomorphism, closed under taking substructures, contains only countably many members up to isomorphism, and has the amalgamation property (see next slide). If these conditions hold, then S is unique up to isomorphism.

If the conditions of the theorem hold, then S is called the Fraïssé limit of the class $\mathcal{C}$.

The amalgamation property

The amalgamation property for a class $\mathcal{C}$ asserts that, if B_1, B_2 are members of $\mathcal{C}$ with a common substructure A, then they can be "glued together" along A to create a larger structure in $\mathcal{C}$.
More formally, if $A, B_1, B_2 \in \mathcal{C}$ and $f_i : A \to B_i$ are embeddings for $i = 1, 2$, then there exists $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ for $i = 1, 2$ so that $f_1 g_1 = f_2 g_2$ (maps written on the right).
For convenience, I allow A to be empty; this special case is the joint embedding property.

The problem

The general problem I will be considering is the following:

Problem

Let S be a homogeneous structure on a countable set X. For which countable groups G is it true that S is a Cayley object for G?

Note that, if all 1-element substructures of S are isomorphic, then $\mathrm{Aut}(S)$ acts transitively on S; if this condition is not satisfied, then there is no possibility for S to be a Cayley object! Thus, for example, we should consider only loopless graphs.

Some structures

In the remainder of this talk, I will say something about the following homogeneous structures:

- the Erdős–Rényi random graph (aka the Rado graph);
- Henson's K_n-free graphs;
- the Urysohn space;
- $\mathbb{Q}$ (as ordered set);
- the generic multiorder (n independent dense orders on a countable set)

The countable random graph

Our first homogeneous structure is the graph R, the Fraïssé limit of the class of finite graphs. It has many remarkable properties. For example, Erdős and Rényi showed that, if we take a countable vertex set and join pairs of vertices independently with probability p (where $0 < p < 1$), the resulting graph is isomorphic to R with probability 1.

The graph R is the most prolific homogeneous Cayley object known, in terms of being a Cayley object for many different groups.

In a group G, a square-root set is a set of the form

$$\sqrt{a} = \{x \in G : x^2 = a\}.$$

It is non-principal if $a \neq 1$.

Theorem

Let G be a countable group which cannot be written as the union of finitely many translates of non-principal square-root sets together with a finite set. Then R is a Cayley object for G.
Indeed, a random Cayley graph for G is isomorphic to R with probability 1.

There is a necessary condition for R to be a Cayley object for a group G, which is formally a tiny bit stronger than the condition of the theorem (though no example of a group satisfying one but not the other is known).
In particular, R is a Cayley graph for the countable abelian group G if and only if either G is an elementary abelian 2-group, or the subgroup of elements of order 1 or 2 has infinite index in G.

An example

The random graph is a Cayley graph for the infinite cyclic group $\mathbb{Z}$. For a set S of positive integers, $\mathrm{Cay}(\mathbb{Z}, S \cup (-S))$ is isomorphic to R if and only if the characteristic function of S is a universal sequence, that is, contains every finite binary sequence as a consecutive subsequence.

A universal sequence can be constructed simply by concatenating the base-2 representations of the natural numbers. This gives an explicit construction for R.

Since there are $2^{\aleph_0}$ universal sequences, it follows that R has $2^{\aleph_0}$ pairwise non-conjugate cyclic automorphisms.

B-groups

A group X is said to be a B-group if every primitive permutation group G containing the regular representation of X is doubly transitive; in other words, if we adjoin permutations to kill all X-invariant equivalence relations (i.e. partitions into cosets of subgroups of X), then we necessarily kill all X-invariant binary relations.

Here B stands for Burnside, who showed (in this terminology) that cyclic groups of prime-power, non-prime, order are B-groups. Much more is known about finite groups now; for example, for almost all n, every group of order n is a B-group.

Problem

Are there any infinite B-groups?

Countable B-groups

If the random graph is a Cayley graph for the countable group X, then X is not a B-group. For the automorphism group of R is primitive but not 2-transitive, and contains the regular action of X.
This simple observation accounts for almost all examples of non-B-groups.
No example of a countably infinite B-group is currently known!

Theorem

There is no countable abelian B-group.

For if the subgroup of involutions has infinite index in A, then R is a Cayley graph for A; otherwise, A has finite exponent, so $A = B \times C$ for some infinite subgroups B and C, and A is a Cayley graph for the countably infinite “rook graph”.

Henson's graphs

The class of finite graphs containing no complete graph of order n is a Fraïssé class. (If we add no new edges when we amalgamate, we cannot create a K_n which was not already there.) Its Fraïssé limit is Henson's graph H_n.

The graph H_3 is a Cayley graph for many countable groups, although the conditions are more restrictive than those for R.

One feature is that probabilistic methods no longer work – a random triangle-free graph has a very strong tendency to be bipartite – and we have to use methods of Baire category instead.

Recently Petrov and Vershik found an exchangeable measure on countable triangle-free graphs which is concentrated on the isomorphism class of Henson's graph (the analogue of Erdős–Rényi); but I have not been able to use their methods to study H_3 as Cayley object.

The situation for H_n for $n > 3$ is quite different.
A slight modification of an argument in Henson's original paper shows:

Theorem

For $n > 3$, the graph H_n is not a normal Cayley graph for any countable group; in particular, it is not a Cayley graph for any abelian group.

Problem

For $n > 3$, is H_n a Cayley graph for any countable group?

Urysohn space

P. S. Urysohn died in 1924 at the age of 26. His last (posthumous) paper was a construction of the unique complete separable metric space which is universal and homogeneous for finite metric spaces. This space is now known as Urysohn space $\mathbb{U}$. His work preceded Fraïssé's, which in its turn preceded that of Erdős and Rényi.

It can be constructed as follows. Define a metric space to be rational if all distances are rational numbers. Now the class of finite rational metric spaces is a Fraïssé class; so it has a Fraïss'é limit, a countable homogeneous universal rational metric space (the so-called rational Urysohn space $\mathbb{U}_{\mathbb{Q}}$). Now $\mathbb{U}$ is the completion of this space.

If $\mathbb{U}_{\mathbb{Q}}$ is a Cayley metric space for an abelian group A, then the action of A extends to $\mathbb{U}$, and the latter is a Cayley metric space for the group $\overline{A}$, the closure of A in the isometry group of $\mathbb{U}$.

Vershik and I showed that there are *many* isometries of $\mathbb{U}$ whose orbits are dense; so the closures of these infinite cyclic groups give *many* abelian groups for which Urysohn space is a Cayley object (in other words, *many* abelian group structures on $\mathbb{U}$).

Unfortunately, we know very little about what these groups look like; in particular, what torsion is possible.

We also showed that $\mathbb{U}_{\mathbb{Q}}$ is a Cayley object for the countable elementary abelian 3-group; so $\mathbb{U}$ is a Cayley object for an uncountable elementary abelian 3-group.

Linear orders

A linear order is a Cayley object for a group G if and only if G is right-orderable, that is, there is an order on G which is invariant under right multiplication.
If P denotes the set of positive elements (greater than the identity), then

- G is the disjoint union of P, $\{1\}$, and P^{-1};
- $P^2 \subseteq P$ (that is, P is a sub-semigroup of G).

Conversely, a set satisfying these two conditions gives rise to a right order on G by the rule $x < y$ if and only if $yx^{-1} \in P$.

Cantor's famous theorem characterises $\mathbb{Q}$ (as ordered set) by the properties that it is dense and without endpoints. It is easy to see that, in this special case, Cantor's conditions are equivalent to Fraïssé's, that is, $\mathbb{Q}$ is the unique countable homogeneous universal ordered set.
Now it is easy to see that a right order on a group G has no endpoints, and it is dense if and only if the set P of positive elements satisfies a stronger version of the second condition above, namely

- $P^2 = P$.

There has been a lot of work on right-ordered groups, and many examples are known. Usually the question of whether the order is dense does not arise, but in many cases it is:

- $\mathbb{Z}$ has no dense right order, but $\mathbb{Z}^2$ has many. For any irrational number α, let $P = \{(x, y) \in \mathbb{Z}^2 : x + \alpha y > 0\}$.
- $\mathbb{Q}$ has a dense right order.
- A non-abelian free group has a dense right order.

Generic multiorders

A *n*-order is a set carrying n total orders. If we don't want to specify n, we speak of a multiorder.
Clearly finite n-orders form a Fraïssé class, so there is a unique countable homogeneous universal n-order, which we call the generic n-order.
It is characterised by the property

Theorem

Let $(X, <_1, \ldots, <_n)$ be an n-order. It is generic if and only if, given any intervals I_i in $(X, <_i)$ (possibly semi-infinite) for $i = 1, \ldots, n$, we have

$$\bigcap_{i=1}^{n} I_i \neq \emptyset.$$

In other words, the intersection of intervals in all but one of the orders is dense in the remaining order.

Permutation patterns

The subject of permutation patterns is connected with 2-orders.
A permutation consists of the numbers $1, 2, \ldots, n$ written in some order. A permutation σ occurs in a permutation π if there is a subset of the positions of π whose entries come in the same relative order as the elements of σ: for example, the highlighted positions show that $\sigma = 132$ occurs in $\pi = 241563$.
Now a permutation is just a finite 2-order (the first order establishes a bijection from the underlying set to $\{1, 2, \ldots, n\}$, while the second order corresponds to the rearrangement of these values given by the permutation). It is easily checked that σ occurs in π if and only if the 2-order corresponding to σ is embeddable as an induced substructure in the 2-order corresponding to π.
Thus, the theory of permutation patterns is the theory of the age of the generic 2-order!

Ramsey classes and extreme amenability

Böttcher and Foniok showed that the class of finite 2-orders is a Ramsey class. This was extended to n-orders by Sokić and by Bodirsky.

According to the beautiful theorem of Kechris, Pestov and Todorcević, if the objects in a Ramsey class have a total order as part of their structure, then the automorphism group of the corresponding countable homogeneous structure is extremely amenable; that is, any continuous action on a compact space has a fixed point.

Thus, automorphism groups of generic multiorders give examples of extremely amenable groups.

Generic multiorders as Cayley objects

The main theorem I want to present here is the following.

Theorem

The generic n-order is a Cayley object for the free abelian group $\mathbb{Z}^m$ if and only if $m > n$.

I will say a bit about the proof of this theorem, since the techniques used (easy diophantine approximation) are perhaps a little unexpected.

The object obtained by deleting some of the orders in a generic multiorder is still generic, and admits any group which the original multiorder admits.

So, in order to prove the theorem, we have to show that the generic n-order is a Cayley object for $\mathbb{Z}^{n+1}$ but not for $\mathbb{Z}^n$.

We saw earlier the case $n = 1$.

The positive direction

The proof in the positive direction uses an important result of Kronecker on diophantine approximation, for which several proofs are given in Chapter XXIII of Hardy and Wright.

Theorem

Let m be a positive integer, and let $c \in \mathbb{R}^m$ be a vector whose components are linearly independent over $\mathbb{Q}$. Then, given any $\epsilon > 0$, any line in $\mathbb{R}^m$ with direction vector c passes within distance ϵ of some lattice point in $\mathbb{Z}^m$.

We also need the existence of a certain kind of matrix. These matrices are plentiful in an abstract (Baire category) sense, though it is not quite straightforward to construct a particular example.

Lemma

Let m be a positive integer. Then there exists a $m \times m$ real matrix A having the properties

- *A is invertible;*
- *each row of A has components which are linearly independent over $\mathbb{Q}$;*
- *the last row of A is orthogonal to all the others.*

Now we give the construction. Define orders $<_1, \ldots, <_{m-1}$ on $\mathbb{Z}^m$ by $x <_i y$ if $a_i.x < a_i.y$, where a_i is the ith row of A, $1 \leq i \leq n = m - 1$. Now an interval in $<_i$ consists of vectors lying between two parallel hyperplanes; the intersection of these intervals is a cylinder with parallelepiped cross-section in a direction orthogonal to the first $m - 1$ rows of the matrix, hence parallel to the mth row. By Kronecker's Theorem, there is a lattice point arbitrarily close to this line, and in particular close enough that it lies in the cylinder defined by the intervals. So this intersection is non-empty in the lattice $\mathbb{Z}^m$.

The negative direction

I will not give so much detail here. Take n orders $<_1, \ldots, <_n$ on $\mathbb{Z}^n$. A theorem of Hölder shows that there are nonzero vectors $c_1, \ldots, c_n$ such that, if $c_i.x < c_i.y$, then $x <_i y$. We now divide into three cases.

- If $c_1, \ldots, c_n$ are linearly independent, and all have components which are linearly independent over $\mathbb{Q}$, then the intersection of intervals is an arbitrarily small parallelepiped, which may contain no lattice point.
- If $c_1, \ldots, c_n$ are linearly independent, and the components of c_i are linearly dependent over $\mathbb{Q}$, then the subspace orthogonal to c_i is free abelian of rank at most $n-1$ with an $n-1$-order; by induction, this $n-1$-order is not generic.
- If $c_1, \ldots, c_n$ are not linearly independent, then some order $<_i$ is determined by the others, so the multiorder is not generic.

Other groups

Having the generic n-order as a Cayley object for $n > 1$ appears to be a much stronger restriction on a countable group than simply having a dense right order.

Problem

For which other groups is the generic n-order is a Cayley object? In particular, are there any such groups which are finitely generated?