The random graph revisited

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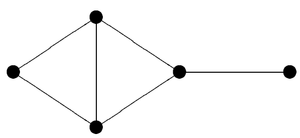
These slides are an amalgamation of those I used for two talks in July 2000, to the Australian Mathematical Society winter meeting in Brisbane, and to the European Congress of Mathematics in Barcelona.

Random graphs

Graph: Vertices, edges; no loops or multiple edges.

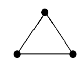
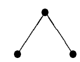
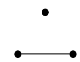
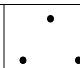


Random: Choose edges independently with probability $1/2$ from all pairs of vertices. (That is, toss a fair coin: Heads = edge, Tails = no edge.)

Random graphs

For *finite* random graphs on n vertices,

- every graph on n vertices occurs with non-zero probability;
- the more symmetric the graph, the smaller the probability.

Graph				
Prob.	$1/8$	$3/8$	$3/8$	$1/8$
$\lvert\mathrm{Aut}\rvert$	6	2	2	6

For *infinite* graphs, the picture is very different . . .

The random graph

Theorem 1 (Erdős and Rényi) *There is a countable graph R with the property that a random countable graph (edges chosen independently with probability $\frac{1}{2}$) is almost surely isomorphic to R.*

The graph R has the properties that

- it is *universal*: any finite (or countable) graph is embeddable as an induced subgraph of R;
- it is *homogeneous*: any isomorphism between finite induced subgraphs of R extends to an automorphism of R.

Sketch proof

Property ($*$) Given finite disjoint sets U,V of vertices, there is a vertex joined to everything in U and to nothing in V.

Step 1 With probability 1, a countable random graph has property ($*$).

Uses the fact that a countable union of null sets is null.

Step 2 Any two countable graphs with property ($*$) are isomorphic.

A standard 'back-and-forth' argument.

Constructions of R

Construction 1. Take any countable model of ZF, and join x to y if $x \in y$ or $y \in x$.

In fact we don't need all of ZF, only the null set, pairing, union, and foundation axioms. So the standard model of finite set theory (the set $\mathbf{N}$, with $x \in y$ if the xth binary digit of y is 1) gives an explicit construction (Rado).

Construction 2. Let $\mathbb{P}_1$ be the set of primes congruent to 1 mod 4. Join p to q if p is a quadratic residue mod q.

If we use instead $\mathbb{P}_{-1}$, the set of primes congruent to -1 mod 4, we obtain the random tournament.

Universal homogeneous structures

Theorem 2 (Fraïssé) *R is the unique countable universal homogeneous graph.*

There are many other examples to which Fraïssé's theorem or variants apply:

- the random tournament, digraph, hypergraph, etc.;
- the universal total order $\mathbb{Q}$ (Cantor), partial order, etc.;
- the universal triangle-free graph (Henson), N-free graph (Covington), locally transitive tournament (Lachlan), two-graph, etc.;
- the universal locally finite group (Hall), Steiner triple system (Thomas), etc.

Measure and category

Measure theory and topology provide two concepts for saying that a set A takes up 'almost all' of the sample space: it may be *of full measure* (the complement of a null set) or *residual* (the complement of a meagre or first category set).

Sometimes these concepts agree (e.g. the random graph is 'ubiquitous' in both senses), sometimes they don't (e.g. Henson's universal triangle-free graph is residual, but a random triangle-free graph is almost surely bipartite).

More generally . . .

There is a more general and powerful version due to Hrushovski. It constructs pseudoplanes, distance-transitive graphs, and examples related to sparse random graphs (among other things).

See the survey article by Wagner in Kaye and Macpherson, *Automorphisms of First-Order Structures*.

Homogeneous graphs

Theorem 3 (Lachlan and Woodrow) *The countably infinite homogeneous graphs are the following:*

(a) the disjoint union of m complete graphs of size n, where m and n are finite or countable (and at least one is infinite);

(b) the complements of the graphs under (a);

(c) the Fraïssé limit of the class of graphs containing no complete subgraph of size r, for given finite $r \geq 3$;

(d) the complements of the graphs under (c);

(e) the random graph (the Fraïssé limit of the class of all finite graphs).

Homogeneous digraphs

Cherlin has determined all the countable homogeneous directed graphs. There are uncountably many analogues of (c), but instead of excluding one complete graph we have to exclude an arbitrary antichain of tournaments. (The examples are due to Henson.)

There are also a few sporadic ones. For example, there are just three homogeneous tournaments: the linearly ordered set $\mathbb{Q}$, the coutable ‘local order’, and the random tournament.

First-order graph properties

The graph R ‘controls’ first-order properties of finite random graphs.

Theorem 4 *(*Glebskii *et al.) A first-order sentence in the language of graphs holds in almost all finite graphs if and only if it holds in R.*

In particular, there is a zero-one law for first-order sentences.

Of course, most interesting graph properties are not first-order!

Indestructibility

R is unchanged by the following operations:

- deleting finitely many vertices;
- adding or removing finitely many edges;
- complementation (interchanging edges and non-edges);
- switching with respect to a finite set (see later).

In addition, the countable random graph with any given edge-probability p satisfying $0 < p < 1$ is isomorphic to R.

Pigeonhole property

A structure X has the *pigeonhole property* if, whenever X is partitioned into two parts, one of the parts is isomorphic to X.

Theorem 5 *The countable graphs with the pigeonhole property are the complete graph, the null graph, and the random graph.*

Theorem 6 (Bonato–Delic) *The countable tournaments with the pigeonhole property are the random tournament, ordinal powers of ω, and their converses.*

Switching

The operation of *switching* a graph Γ with respect to a set X of vertices, as defined by Seidel, works as follows: interchange edges and non-edges between X and its complement, leaving edges within and outside X unaltered.

Let $\mathcal{T}(\Gamma)$ be the set of triples of vertices of Γ containing an odd number of edges.

Theorem 7 *Graphs Γ_1 and Γ_2 on the same vertex set are related by switching if and only if $\mathcal{T}(\Gamma_1) = \mathcal{T}(\Gamma_2)$.*

In different language, this says that

$$H^2(\text{simplex}, \mathbb{Z}/2\mathbb{Z}) = 0.$$

Switching has many applications in finite and Euclidean geometry, group theory, strongly regular graphs, etc.

Reducts

A subgroup of the symmetric group is *closed* in the topology of pointwise convergence if and only if it is the automorphism group of a first-order structure (which can be taken to be a homogeneous relational structure).

Theorem 8 (Thomas) *There are five closed subgroups of $\mathrm{Sym}(R)$ containing $\mathrm{Aut}(R)$, viz. $\mathrm{Aut}(R)$, the group of automorphisms and anti-automorphisms of R, the group of switching-automorphisms of R, the group of switching-automorphisms and anti-automorphisms of R, and $\mathrm{Sym}(R)$.*

Similar results are known in a few other cases, e.g. $\mathbb{Q}$ (as ordered set), random hypergraphs.

Reconstruction

The *perfect graph theorem* (Lóvasz) asserts that the complement of a perfect graph is perfect.

The *P_4-structure* of a graph is the collection of subsets which induce paths on four vertices. The *semi-strong perfect graph theorem* (Reed) asserts that a graph which has the same P_4-structure as a perfect graph is perfect.

Cameron and Martins proved that for almost all finite graphs G, the only graphs with the same P_4-structure as G are G and its complement. An analogous statement is true with any finite collection $\mathcal{F}$ of finite graphs in place of $\{P_4\}$: for almost all finite graphs Γ, any graph having the same $\mathcal{F}$-structure as Γ is related to it by one of five equivalence relations corresponding to Thomas' five reducts. (We saw this already for switching, where $\mathcal{F}$ is the class of graphs with three vertices and an odd number of edges.)

The proof uses Theorems 8 and 4, and some elementary model theory.

Cyclic automorphisms

Let σ be a cyclic automorphism of the countable graph Γ, permuting the vertices in a single cycle. Then we can label the vertices with $\mathbb{Z}$, so that σ is the cyclic shift $x \mapsto x+1$.

Let $S = \{x > 0 : x \sim 0\}$. Then S determines

- Γ up to isomorphism;
- σ up to conjugacy in $\mathrm{Aut}(\Gamma)$.

Now for almost all random choices of S, we find that Γ is isomorphic to R. As a corollary, we see immedately that R has $2^{\aleph_0}$ conjugacy classes of cyclic automorphisms.

Cycle structure

Truss gave a characterisation of all cycle structures of automorphisms of R. For example, R admits a cyclic automorphism, and an automorphism which fixes a vertex v and has two infinite cycles on the remaining vertices (the neighbours and non-neighbours of v, respectively).

The following curious property holds:

Note If a permutation g of a countable set leaves some copy of R invariant, then the probability that a random g-invariant graph is isomorphic to R is strictly positive.

A random g-invariant graph is obtained by deciding independently whether each orbit of $\langle g \rangle$ on 2-sets consists of edges or non-edges.

It is not known whether the analogous property for arbitrary permutation groups on a countable set is true or false.

Generic automorphisms

An element g of a group G is *generic* if the conjugacy class g^G containing g is residual in G. (We have to use Baire category here since there is no natural measure on the infinite symmetric group.)

Truss showed that R admits generic automorphisms. These have infinitely many cycles of each finite length but (surprisingly) no infinite cycles.

Hodges, Hodkinson, Lascar and Shelah showed that R admits generic n-tuples of automorphisms (that is, $\mathrm{Aut}(R)^n$ has generic elements) for every positive integer n. If $(g_1, \ldots, g_n)$ is generic, then $\langle g_1, \ldots, g_n \rangle$ is a free group of rank n, all of whose orbits on R are finite.

On the next slide we have the other extreme ...

Other free subgroups

Theorem 9 (Bhattacharjee and Macpherson) *There exist automorphisms f,g of R such that*

(a) f,g generate a free subgroup of $\mathrm{Aut}(R)$,

(b) f has a single cycle on R, which is infinite,

(c) g fixes a vertex v and has two cycles on the remaining vertices (namely, the neighbours and non-neighbours of v),

(d) the group $\langle f,g\rangle$ is oligomorphic, and transitive on vertices, edges, and non-edges of R, and each of its non-identity elements has only finitely many cycles on R.

A permutation group is *oligomorphic* if the automorphism group has only finitely many orbits on n-tuples for all natural numbers n.

B-groups

A *B-group* is a group X with the property that any primitive permutation group G which contains the right regular action of X is doubly transitive.

Theorem 10 *For almost all n, every group of order n is a B-group.*

The proof uses the Classification of Finite Simple Groups.

By contrast, no countable B-groups are currently known.

Countable non-B-groups

A *square-root set* in a group X is a set of the form

$$\sqrt{a} = \{x : x^2 = a\}.$$

It is *non-principal* if $a \neq 1$.

Theorem 11 (Cameron and Johnson) *Suppose that the countable group X is not the union of finitely many translates of non-principal square-root sets. Then a random Cayley graph for X is isomorphic to R. Hence X is not a B-group.*

In particular, R is a Cayley graph for the infinite cyclic group $\mathbb{Z}$. This gives another proof that R admits cyclic automorphisms.

Countable non-B-groups

Theorem 12 *There is no countable abelian B-group.*

Proof. Let $X_2 = \{x \in X : x^2 = 1\}$. If $|X : X_2|$ is infinite, then Theorem 11 applies. Otherwise, X has finite exponent and so $X = Y \times Z$ with Y and Z infinite; then X is contained in the primitive group $\mathrm{Sym}(\mathbf{N}) \wr \mathrm{Sym}(2)$ (with the product action).

Theorem 13 *A countable simple group with more than two conjugacy classes is not a B-group.*

Proof. Consider $\{x \mapsto a^{-1}xb : a,b \in X\}$.

Problems on B-groups

- Can one use other countable homogeneous structures to find further non-B-groups? What about Hrushovski's method?
- Are there any countable B-groups?
- Is the hypothesis of Theorem 11 necessary?

The small index property

A countable structure M has the *small index property* if any subgroup of index smaller than $2^{\aleph_0}$ in $\mathrm{Aut}(M)$ contains the pointwise stabiliser of a finite set of points.

If M has the small index property, then the topology on $\mathrm{Aut}(M)$ (induced by the topology of pointwise convergence in the symmetric group) is determined by the group structure: a subgroup is open if and only if it has index less than $2^{\aleph_0}$.

A countable structure M has the *strong small index property* if any subgroup of index smaller than $2^{\aleph_0}$ in $\mathrm{Aut}(M)$ lies between the pointwise and setwise stabiliser of a finite set of points.

Small index property for R

Theorem 14 *The random graph has the (strong) small index property.*

The small index property was shown by Hodges, Hodkinson, Lascar and Shelah, who showed that $\mathrm{Aut}(R)$ has generic n-tuples of elements (i.e. there is a single conjugacy class which is comeagre in $\mathrm{Aut}(R)$).

Corollary $\mathrm{Aut}(R)$ *is not isomorphic to the automorphism group of any other countable homogeneous graph or digraph.*

Indeed, $\mathrm{Aut}(R)$ cannot act transitively on vertices, edges and non-edges on any other countable graph or digraph except R.

Footnote: topology

After the Barcelona talk, A. Vershik drew my attention to the universal homogeneous metric space constructed by P. S. Urysohn in his last paper. There should be interesting parallels to be drawn here!

A. M. Vershik, The universal Urysohn space, Gromov metric triples and random metrics on the natural numbers, *Russian Math. Surveys* **53** (1998), 921–928.

See also the work of Neumann on the 'rational world', the countable 0-dimensional space without isolated points (realised as $\mathbb{Q}$, and characterised by Sierpiński).

P. M. Neumann, Automorphisms of the rational world, *J. London Math. Soc.* (2) **32** (1985), 439–448.

More on this later, hopefully.

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