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Counting, structure, and symmetry

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Some counting problems

Count functions $f : X \to C$ with $|X| = n$, $|C| = k$ (that is, colourings of X with k colours), subject to some combination of structure and symmetry on X and C, as follows:

- a graph Γ on X, with f a proper colouring;
- also a graph Γ' on C, with f a homomorphism;
- groups G and G' acting on X and C (as automorphisms of the graphs if present), count up to the group action (that is, count orbits).

Here G acts by $f^g(x) = f(x^{g^{-1}})$, and G' acts by $f^{g'}(x) = (f(x))^{g'}$.

These are only examples; many other interpretations of "structure" are possible!

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Examples

For the four combinations of Γ the null or complete graph, G the trivial or symmetric group, we obtain the counts for sampling with or without replacement, with ordered or unordered samples. So the answers to the four counting problems are respectively k^n, $k(k-1)\cdots(k-n+1)$, $\binom{k+n-1}{n}$, and $\binom{k}{n}$, respectively.

If $k \geq n$ and we take G' to be the symmetric group, we obtain the Bell number $B(n)$ if G is the trivial group, and the partition number $p(n)$ if G is the symmetric group.

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Structure on X

Let's just consider the case where we put structure only on X.

- If there is no structure on X, the number is k^n.
- If the is a graph Γ on X, the number is $P_\Gamma(k)$ (the chromatic polynomial of Γ evaluated at k), a polynomial in k with leading term k^n.

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Symmetry on X

- If there is a group G on X, the number of orbits is

$$\frac{1}{|G|}\sum_{g\in G} k^{c(g)}$$

(where $c(g)$ is the number of cycles of g), a polynomial with leading term $k^n/|G|$. This follows from the Orbit-Counting Lemma, since g fixes $k^{c(g)}$ colourings.

- If we have both graph and group, the number is again a polynomial with leading term $k^n/|G|$. For if two vertices in a cycle of g are adjacent, then g fixes no colourings; otherwise it fixes $P_{\Gamma^g}(k)$ colourings, where Γ^g is obtained by shrinking each cycle of g to a single vertex.

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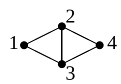
Example

Let Γ be the following graph, and let G be the group whose elements are the identity, $(1,4)$, $(2,3)$, and $(1,4)(2,3)$.



The chromatic polynomial of Γ is $k(k-1)k-2)^2$. The automorphisms $(2,3)$ and $(1,4)(2,3)$ fix no colourings, whereas $(1,4)$ fixes $(1,4)$ fixes $k(k-1)(k-2)$ colourings, since the graph $\Gamma^{(1,4)}$ is a triangle. So the number of orbits is

$$\tfrac{1}{4}k(k-1)^2(k-2).$$

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Nowhere-zero flows

Let A be an abelian group of order k. A *flow* on a graph Γ with values in A is defined as follows. Take an (arbitrary but fixed) orientation of the edges of Γ. Now a flow is a function from the set of oriented edges to A such that, at each vertex v, the total flow into and out of v are equal (the sums computed in A). It is *nowhere zero* if it doesn't take the value $0 \in A$. If a graph has a bridge, then it nas no nowhere-zero flows; so we assume for the time being that our graphs are bridgeless.

It is known that the number of nowhere zero flows on Γ with values in A depends only on Γ and the order k of A, not on the detailed structure of A. Moreover, this number is a polynomial in k, with leading coefficient 1.

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Orbits on nowhere-zero flows

If G is a group of automorphisms of Γ, then G acts on the set of nowhere-zero flows on Γ in a natural way. (An automorphism may change the orientation of an edge; if so, we require that it should negate the value of the flow on that edge.)

Bill Jackson considered the case where $A = C_2^m$, so that $k = 2^m$. In this case, every element is equal to its inverse, so we don't have to worry about this problem. He showed that, in this case, the number of G-orbits on nowhere-zero flows is a polynomial in k, whose leading coefficient is $1/|\bar{G}|$, where $\bar{G}$ is a certain factor group of G.

As the next example shows, in general the answer does depend on the structure of A, not just its order.

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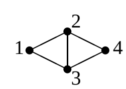
An example



A nowhere-zero flow takes values a on 23, b on 21 and 13, and c on 24 and 43, with $a+b+c=0$. We can choose any non-zero a, and any $b \neq -a$; then $c = -a-b$. So there are $(k-1)(k-2)$ n.z. flows.

A flow is fixed by $(2,3)$ if and only if $2a = 2b = 2c = 0$. So the number of such flows is $(\alpha_2 - 1)(\alpha_2 - 2)$, where α_2 is the number of solutions of $2x = 0$ in A.

A flow is fixed by $(1,4)$ if and only if $a = b$. So $c = -2a$, whence there are $k = \alpha_2$ choices for the flow.

Finally, a flow fixed by $(1,4)(2,3)$ must vanish on 23. So by the Orbit-Counting Lemma, the number of orbits on n.z. flows is

$$\frac{1}{4}((k-1)(k-2) + (\alpha_2 - 1)(\alpha_2 - 2) + (k - \alpha_2)).$$

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Orbits on nowhere-zero flows, continued

We have found that the following result holds:

Theorem 1 *Let G be a group of automorphisms of a graph Γ. Then there is a polynomial $P(\Gamma,G)$ in indeterminates q_i indexed by non-negative integers i, with the following property:*
Given an abelian group A, the number of G-orbits on nowhere-zero A-flows on Γ is $P(\Gamma,G;q_i \leftarrow \alpha_i)$, where α_i is the number of solutions of the equation $ia = 0$ for $a \in A$.

Note that α_0 is the order of the group A. Moreover, if A is an elementary abelian 2-group, then

$$\alpha_i = \begin{cases} 1 & \text{if } i \text{ is odd,} \\ |A| & \text{if } i \text{ is even,} \end{cases}$$

so we recover Jackson's polynomial.

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First reduction

It suffices to prove that, for any automorphism g of Γ, the number of nowhere-zero A-flows fixed by g is of the form $p(\Gamma,g;q_i \leftarrow \alpha_i)$, for some polynomial $p(\Gamma,g)$.
For, by the Orbit-Counting Lemma, the number of orbits of a group is the average number of fixed points of its elements: thus

$$P(\Gamma,G) = \frac{1}{|G|}\sum_{g\in G} p(\Gamma,g).$$

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Second reduction

It suffices to prove that, for any automorphism g of Γ, the total number of A-flows fixed by g is of the form $p^*(\Gamma,g;q_i \leftarrow \alpha_i)$, for some polynomial $p^*(\Gamma,g)$.
For, if I indexes the set of cycles of g on edges of Γ, and $\Gamma(J)$ is obtained from Γ by deleting edges in cycles indexed by J, then Inclusion-Exclusion gives

$$p(\Gamma,g) = \sum_{J\subseteq I}(-1)^{|J|}p^*(\Gamma(J),g),$$

since $p^*(\Gamma(J),g;q_i \leftarrow \alpha_i)$ is the number of flows fixed by g which vanish at least on the edges in orbits with indices in J.

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The final step

Let M be the vertex-edge incidence matrix of Γ (with respect to a given orientation). Then an A-flow on Γ is a vector f (with components in A) satisfying $Mf = 0$.
Now let M_g be obtained from M by adding, for each pair (e_i,e_j) of edges in the same cycle of g, a row with ith entry 1, jth entry -1 if e_i^g and e_j have the same orientation and $+1$ otherwise, and other entries 0.
Then f is an A-flow fixed by g if and only if $M_g f = 0$.

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The final step, continued

By elementary row and column operations (which don't change the number of solutions in any given abelian group A), we can convert M_g to Smith normal form, with (i,i) entry d_i for $i \leq r$ and all other entries zero, and r is the rank of M_g. Now the first r equations are $d_i x_i = 0$ (which has α_{d_i} solutions in A) and the last $m-r$ are trivial (and have $|A| = \alpha_0$ solutions).

So the number of solutions is $p^*(\Gamma, g; q_i \leftarrow \alpha_i)$, where

$$p^*(\Gamma, g) = \left(\prod_{i=1}^{r} q_{d_i}\right) q_0^{m-r}.$$

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Calculating α_i

For any abelian group A, we have $\alpha_0 = |A|$ and $\alpha_1 = 1$.

In general, A is a direct sum of cyclic groups, say

$$A = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r};$$

then we have

$$\begin{aligned} \alpha_i(A) &= \alpha_i(C_{n_1}) \cdot \alpha_i(C_{n_2}) \cdots \alpha_i(C_{n_r}) \\ &= \gcd(i, n_1) \cdot \gcd(i, n_2) \cdots \gcd(i, n_r). \end{aligned}$$

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Where next?

The method we have used for nowhere-zero flows extends to nowhere-zero tensions in graphs, and to words of given weight in linear codes.

We would like to extend the method to any counting problem whose solution (without the group action) is given by a specialisation of the Tutte polynomial.

We would also like to replace the use of the Orbit-Counting Lemma by the Cycle Index Theorem.