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The Rado graph and the Urysohn space

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The countable random graph

We begin with the *countable random graph* or *Rado graph R*.
A graph G is *homogeneous* if every isomorphism betweem (finite) induced subgraphs of G extends to an automorphism of G.
This is a very strong symmetry condition on a graph. In particular, a homogeneous graph is vertex-transitive, edge-transitive, non-edge-transitive, ...
A countable graph is *universal* if every (at most) countable graph can be embedded in it as an induced subgraph.

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The random graph

Theorem 1 (Erdős and Rényi) *There is a countable graph R with the property that a random countable graph (edges chosen independently with probability $\frac{1}{2}$) is almost surely isomorphic to R.*

The graph R has the properties that

- it is *universal*: any finite (or countable) graph is embeddable as an induced subgraph of R;
- it is *homogeneous*: any isomorphism between finite induced subgraphs of R extends to an automorphism of R.

As well as being the "random graph", R is also generic in the sense of Baire category (with respect to a natural metric on the set of all graphs on a fixed countable vertex set).

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Sketch proof

Property ($*$) Given finite disjoint sets U,V of vertices, there is a vertex joined to everything in U and to nothing in V.

Step 1 With probability 1, a countable random graph has property ($*$).
Calculation shows that, for a fixed pair U, V, the probability that no such vertex z exists is zero. Then use the fact that a countable union of null sets is null.

Step 2 Any two countable graphs with property ($*$) are isomorphic.
A standard 'back-and-forth' argument: condition ($*$) allows us to extend any partial isomorphism (in either direction) to any further point.

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Explicit constructions

The argument by Erdős and Rényi is a non-constructive existence proof, and they offered no explicit construction.

The year after the paper by Erdős and Rényi, an explicit construction for R was given by Rado (though apparently without noticing that it was the random graph. The vertex set is the set of natural numbers; for $x < y$, we join x to y if the xth digit of y (written in base 2) is one. (The joining rule is symmetric.)

Two other constructions:

- Vertices are primes congruent to 1 mod 4; join p to q if p is a square mod q (this is symmetric by quadratic reciprocity).
- Take a countable model of the Zermelo–Fraenkel axioms for set theory, and symmetrise the membership relation.

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Automorphisms of R

The homogeneous graph R has a very rich automorphism group. Here are some of its properties.

- (Truss) $\mathrm{Aut}(R)$ is simple and has cardinality $2^{\aleph_0}$.
- (Cameron–Johnson) $\mathrm{Aut}(R)$ contains $2^{\aleph_0}$ conjugacy classes of cyclic automorphisms.
- (Truss) $\mathrm{Aut}(R)$ contains generic elements (that is, a conjugacy class which is residual in $\mathrm{Aut}(R)$ in the sense of Baire category). All cycles of such elements are finite, but they have infinite order.

Truss also found all possible cycle structures of automorphisms of R.

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Automorphism groups of R

A body of results describe various interesting subgroups of $\mathrm{Aut}(R)$:

- (Hodges, Hodkinson, Lascar, Shelah) $\mathrm{Aut}(R)$ contains generic n-tuples of elements. Any such n-tuple generates a free group of rank n, all of whose orbits are finite.
- (Bhattacharjee, Macpherson) $\mathrm{Aut}(R)$ contains a free group of rank 2 whose non-identity elements have only finitely many cycles.
- (Bhattacharjee, Macpherson) $\mathrm{Aut}(R)$ contains a dense locally finite subgroup.

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Regular automorphism groups and Cayley graphs

A group acts as a regular group of automorphisms of a graph if and only if the graph is a Cayley graph for the group.

The existence of many cyclic automorphisms of R is proved by showing that, with probability 1, a random Cayley graph for the infinite cyclic group is isomorphic to R.

Cameron and Johnson found that, if the countable group X is not the union of finitely many translates of square-root sets of non-identity elements together with a finite set, then a random Cayley graph for X is isomorphic to R with probability 1.

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Countable homogeneous structures

Fraïssé gave a necessary and sufficient condition for a class $\mathscr{C}$ of finite structures to be the finite substructures of a countable homogeneous structure. The most important condition is the *amalgamation property*. If the conditions are satisfied, then the countable structure is unique up to isomorphism, and is called the *Fraïssé limit* of $\mathscr{C}$.

In particular, we have:

Theorem 2 (Fraïssé) *R is the unique countable universal homogeneous graph.*

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Countable homogeneous graphs

Theorem 3 (Lachlan and Woodrow) *The countably infinite homogeneous graphs are the following:*

(a) the disjoint union of m complete graphs of size n, where m and n are finite or countable (and at least one is infinite);

(b) the complement of a graph under (a);

(c) the Henson graph H_n*, the Fraïssé limit of the class of graphs containing no complete subgraph of size r, for given finite $r \geq 3$;*

(d) the complement of a graph under (c);

(e) the random graph (the Fraïssé limit of the class of all finite graphs).

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Regular automorphism groups

Which of the Henson graphs has a regular automorphism group? That is, which is a Cayley graph?
Henson showed that H_3 has cyclic automorphisms but H_r does not for $r > 3$.
More generally, we have:

• H_3 is a Cayley graph for any one of a large class of countable groups (there is a characterisation like that of Cameron and Johnson for R);

• for $r > 3$, H_r is not a normal Cayley graph for any countable group X (that is, there is no graph admitting both the left and the right regular action of X). It is not known whether H_r can be a Cayley graph for $r > 3$.

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Urysohn space

In a posthumous paper published in 1927, P. S. Urysohn showed that there exists a unique universal and homogeneous *Polish space* (complete separable metric space) U. Here "homogeneous" means that any isometry between finite subsets extends to an isometry of the whole space; "universal" means that any Polish space can be isometrically embedded into U.
This result is a precursor of the work of Fraïssé; the separability condition plays the role of countability in Fraïssé's work.
Vershik has shown that U is the random metric space with respect to a wide class of natural measures on the class of Polish spaces, and that it is generic.
We do not yet have a simple explicit description of U.

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Urysohn and Fraïssé

A convenient construction of U is as follows. Let Q be the universal homogeneous "rational metric space": the Fraïssé limit of the class of finite metric spaces with rational distances. Then U is the completion of Q. Moreover, any isometry of Q extends uniquely to an isometry of U.
Our strategy is to build isometry groups of Q using similar techniques to those used for R earlier, they or their closures in $\mathrm{Aut}(U)$ provide us with interesting isometry groups of U.

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Regular automorphisms

There are $2^{\aleph_0}$ non-conjugate cyclic isometries of Q (permuting all vertices in a single cycle). Each of these has the property that all its orbits on U are dense. In particular, the closure of the group generated by such an isometry (in the natural topology on $\mathrm{Aut}(U)$) is an abelian group acting transitively on U.

Problem: What can one say about the structure and conjugacy of the abelian groups arising in this way? Note that these groups are not necessarily torsion-free.

Moreover, one can show that the condition of Cameron and Johnson for a group to act regularly on R also guarantees a regular action on Q. Again one can ask what the closure of such a group looks like.

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Normal structure

An isometry σ whose cycles are dense in U has the property that $d(u,\sigma(u))$ is constant for all points $u \in U$. Hence it lies in the normal subgroup $B(U)$ of $\mathrm{Aut}(U)$ consisting of *bounded isometries*, those for which $d(u,\sigma(u))$ is bounded. Thus this subgroup is non-trivial; it is also easy to see that it is not the whole of $\mathrm{Aut}(U)$ (that is, unbounded isometries exist).

Problem: Is it true that $B(U)$ and $\mathrm{Aut}(U)/B(U)$ are simple?

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A dense free subgroup

Using a trick invented by Tits, we can show:

Theorem 4 *There is a subgroup F of $\mathrm{Aut}(U)$ which is a free group of countable rank and is dense in $\mathrm{Aut}(U)$.*

The proof depends on the facts that

- $\mathrm{Aut}(U)/B(U)$ contains a free subgroup;
- $B(U)$ is a dense subgroup of $\mathrm{Aut}(U)$.

Problem: Does the analogue of Bhattacharjee–Macpherson hold? That is, does $\mathrm{Aut}(U)$ have a dense locally finite subgroup?