Cyclic automorphisms of homogeneous structures

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Homogeneous and universal structures

A structure of some kind is *homogeneous* if any isomorphism between finite (induced) substructures can be extended to a global automorphism of the whole structure.

A srtucture X is *universal* for a class $\mathscr{C}$ if $X \in \mathscr{C}$ and any structure in $\mathscr{C}$ "no larger than X" can be embedded in X.

We are mainly interested in *graphs* and *metric spaces*; we interpret the size bound as countability and separability respectively.

Urysohn and Rado

In a posthumous paper published in 1927, Urysohn showed that there is a homogeneous and universal object $\mathbb{U}$ in the class of complete separable metric spaces, by giving an explicit construction.

The analogous result in the simpler case of countable graphs was not proved until the 1960s. Rado in 1964 constructed a countable universal graph R. It is not clear whether he knew that it is homogeneous (but see comments on Erdős and Rényi below).

Fraïssé

The *age* of a relational structure X is the class of all finite substructures embeddable in X.

Already in the early 1950s, Fraïssé had given necessary and sufficient conditions for a class $\mathscr{C}$ of finite relational structures to be the age of a countable homogeneous structure X. There are three rather trivial "book-keeping" conditions and one important condition: the *amalgamation property*. He showed that, if these conditions are satisfied, then X is unique up to isomorphism. It is called the *Fraïssé limit* of $\mathscr{C}$.

Moreover, the Fraïssé limit of $\mathscr{C}$ is universal for the class of countable structures whose age is contained in $\mathscr{C}$.

Urysohn and Rado through Fraïssé's eyes

The class of all finite graphs satisfies Fraïssé's conditions. Its Fraïssé limit is Rado's graph R.

For Urysohn, things are not so simple. There are too many finite metric spaces for them all to be embeddable in a given countable space. But if we take, for example, the class $\mathscr{C}$ of finite metric spaces with all distances rational, then Fraïssé's conditions are satisfied, so there is a Fraïssé limit $Q\mathbb{U}$. The Urysohn space $\mathbb{U}$ is the completion of $Q\mathbb{U}$.

Slide 6

Erdős–Rényi and Vershik

In 1963 (the year before Rado's paper), Erdős and Rényi "demolished the theory of infinite random graphs" by showing that there is, up to isomorphism, a unique countable random graph (where we choose edges independently with probability $1/2$). They gave no explicit construction (their argument is a non-constructive existence proof); but, not surprisingly, Rado's graph R is the countable "random graph".

The analogous result for complete separable metric spaces was not proved until 2002, when Vershik showed that, for a wide range of "natural" measures, the random metric space is isometric to $\mathbb{U}$.

Similar but easier results hold if we use Baire category instead of measure. Any Fraïssé limit is residual in the class of all countable structures with the same or smaller age (using the natural ultrametric on this set).

Slide 7

Henson, Lachlan–Woodrow and Cherlin

In 1971, Henson observed that, for any $n \geq 3$, the class of K_n-free finite graphs satisfies Fraïssé's conditions, so that there is a unique homogeneous K_n-free graph H_n.

In 1980, Lachlan and Woodrow showed that, up to complementation, and excluding some "trivial examples" (disjoint unions of complete graphs), the only countable homogeneous graphs are Rado's graph R and the Henson graphs H_n.

Henson also constructed uncountably many countable homogeneous directed graps, The classification of all such digraphs was a major piece of work by Cherlin, published in 1998.

Note: H_3 is the generic triangle-free graph, in the sense of Baire category; but the random triangle-free graph is almost surely bipartite.

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Cyclic automorphisms of graphs

If a graph X has a *cyclic automorphism* σ (one which permutes all the vertices in a single cycle), then the vertices of X can be indexed by the integers so that σ is the cyclic shift $x \mapsto x+1$. Let S be the set of positive integers n such that x_n is adjcent to x_0 in X. Now S determines X up to isomorphism, and σ up to conjugacy in $\text{Aut}(X)$. In fact, X is the Cayley graph of the infinite cyclic group with respect to S (strictly, $S \cup -S$).

(The *Cayley graph* $\text{Cay}(G,S)$ of a group G with respect to a set S is the graph with vertex set G, with x joined to y if and only if $xy^{-1} \in S$. The group G acts by right multiplication as a group of automorphisms of the graph.)

It can be shown that, if we choose a random set S of positive integers, then with probability 1, the resulting graph X is isomorphic to R. Hence R has $2^{\aleph_0}$ pairwise non-conjugate cyclic automorphisms.

Henson showed that H_3 also has many cyclic automorphisms, but H_n has no cyclic automorphisms for $n \geq 4$.

Slide 9

Cyclic automorphisms of $Q\mathbb{U}$

$\mathbb{U}$ is uncountable and so cannot have a cyclic automorphism. Nevertheless, its dense subspace $Q\mathbb{U}$ does have cyclic automorphisms, and indeed has uncountably many non-conjugate such. The proof is a little more elaborate than the proof for R; we cannot choose the distances arbitrarily since, for example, $d(x, \sigma^n(x)) \leq nd(x, \sigma(x))$.

Now σ induces an automorphism of $\mathbb{U}$, all of whose orbits are dense. The closure of $\langle \sigma \rangle$ is an abelian group A which acts transitively (and hence regularly) on $\mathbb{U}$. So $\mathbb{U}$ has an abelian group structure (indeed, many such).

Note that different choices of σ give rise to different structures for the group A; this is not well understood yet!

R as a Cayley graph

What if we replace the infinite cyclic group with an arbitrary countable group? Which countable groups *G* act regularly on a given homogeneous graph *X*? In other words, which homogeneous graphs *X* are Cayley graphs for a given countable group *G*?

In the case of Rado's graph, we can write down a somewhat complicated necessary and sufficient condition for a countable group *G* to act regularly on *R*. This condition implies that if *G* cannot be written as the union of finitely many translates of square root sets of non-identity elements together with a finite set, then *G* acts regularly on *R*. In particular, this holds if any non-identity element has only finitely many square roots.

R as a Cayley graph, continued

For example, if *G* is any finite or countable group, the direct product $G \times Z$ acts regularly on *R* (where *Z* is the infinite cyclic group). So Aut(*R*) embeds all countable groups in a very special manner.

An example of a group which does not act regularly on *R* is the infinite *dicyclic group*

$$G = \langle a, b : b^4 = 1, b^{-1}ab = a^{-1} \rangle.$$

In any Cayley graph for *G*, any vertex *x* is either joined to both or neither of 1 and b^2, or to both or neither of *b* and b^3.

Henson's graphs as Cayley graphs

We saw that H_3 admits cyclic automorphisms. Indeed, H_3 admits regular actions of many countable groups. The known sufficient conditions are more stringent than those for *R*, but allow (for example) all abelian groups where the subgroups consisting of elements of orders 2 or 3 are finite.

A *normal Cayley graph* is a Cayley graph Cay(*G*, *S*) for which *G* acts by both left and right multiplication; equivalently, *S* is a normal subset of *G*. Following Henson's argument, it can be shown that, for $n > 3$, the graph H_n is not a *normal Cayley graph* for any countable group.

It is not known whether H_n is a Cayley graph, or whether Aut(H_n) embeds all countable groups.

$Q\mathbb{U}$ as a Cayley object

R is a *reduct* of $Q\mathbb{U}$; that is, there is a partition of the positive rationals into two sets *E* and *N* such that, if we join two points whose distance belongs to *E*, the resulting graph is isomorphic to *R*. (Indeed there are many such partitions). Thus, any group acting regularly on $Q\mathbb{U}$ also acts regularly on *R*.

however, this implication does not reverse. The countable elementary abelian group of exponent 3 does not act regularly on $Q\mathbb{U}$, and indeed, does not act on $\mathbb{U}$ with dense orbits.

On the other hand, the countable elementary abelian group of exponent 2 does act regularly on $Q\mathbb{U}$. Its closure in the isometry group of $\mathbb{U}$ is a regular elementary abelian group of exponent 2.

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Countable B-groups

A group *G* is said to be a *B-group* if every primitive group *H* containing the regular action of *G* is doubly transitive. The B stands for Burnside, who showed that finite cyclic groups of composite order are B-groups. Using the Classification of Finite Simple Groups, it is easy to show that for almost all *n*, every group of order *n* is a B-group.

On the other hand, no countable B-group is known. The most powerful tool for showing that a countable group is not a B-group is Rado's graph, whose automorphism group is primitive but not doubly transitive. So every group acting regularly on *R* fails to be a B-group.

The remarks on the last slide show that $Q\mathbb{U}$ doesn't give us any further examples.

What next?

Here are a couple of further questions in addition to those mentioned above.

Other structures The countable homogeneous total order is $\mathbb{Q}$; a group acts regularly on $\mathbb{Q}$ if and only if it has a dense right order. The infinite cyclic group does not, but many other countable torsion-free groups do, including the free and free abelian groups of rank greater than 1.

There is a very interesting countable homogeneous universal poset *P*. It does not admit the infinite cyclic group: indeed, a countable poset with a cyclic automorphism is either disconnected, or has only finitely many elements incomparable with a given one. Does any group act regularly on *P*?

Homomorphisms What happens if we replace "isomorphism" with "homomorphism" in the definition of homogeneity? Jarik Nešetřil and I have some partial results on this.