Some aspects of codes over rings

Peter J. Cameron
Queen Mary
University of London
p.j.cameron@qmul.ac.uk

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This is work by two of my students,
Josephine Kusuma and Fatma Al-Kharoosi
Summary

- Codes over rings and orthogonal arrays
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- $\mathbb{Z}_4$ codes and Gray map images
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- $\mathbb{Z}_4$ codes and Gray map images
- $\mathbb{Z}_4$ codes determined by two binary codes
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- $\mathbb{Z}_4$ codes and Gray map images
- $\mathbb{Z}_4$ codes determined by two binary codes
- Generalisation to $\mathbb{Z}_{p^n}$
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Codes over rings

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We define the (Hamming) metric $d_H$, the inner product of words, and the dual of a code, over a ring $R$ just as for codes over fields.
Orthogonal arrays

A code $C$ over an alphabet $R$ is an orthogonal array of strength $t$ if, given any set of $t$ coordinates $i_1, \ldots, i_t$, and any entries $r_1, \ldots, r_t \in R$, there is a constant number of codewords $c \in C$ such that $c_{i_k} = r_k$ for $k = 1, \ldots, t$. 
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A theorem

**Theorem**

*The strength of the linear code $C$ over $R$ is one less than the Hamming weight of the dual code $C^\perp$.***
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This was proved by Delsarte for codes over fields. The generalisation is not completely straightforward. It depends on the following property of rings (which, here, mean finite commutative rings with identity).
A theorem about rings

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Now the theorem is the case \( n = 1 \) of the coding result: a code of length 1 is just an ideal of \( R \) and the dual code is its annihilator. The general case is then proved by a careful induction.
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It is not true that $|\text{Ann}(I)| = |R|/|I|$ for any ideal $I$, and hence not true that $|C^\perp| = |R|^n/|C|$ for any code over the ring $R$. However this does hold for rings such as the integers mod $q$ for positive integers $q$, or for finite fields.
The Gray map

The **Lee metric** $d_L$ on $\mathbb{Z}_4^n$ is defined coordinatewise:

$$d_L(v, w) = \sum_{i=1}^{n} d_L(v_i, w_i),$$

where the Lee metric on $\mathbb{Z}_4$ is given by the rule that $d_L(a, b)$ is the number of steps from $a$ to $b$ when the elements of $\mathbb{Z}_4$ are arranged round a circle.

It was introduced by Hammons et al. in their classic paper showing that certain nonlinear binary codes such as the Nordstrom–Robinson, Preparata and Kerdock codes are Gray map images of linear $\mathbb{Z}_4$-codes.
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The **Gray map** \(\gamma\) is a non-linear map from \(\mathbb{Z}_4^n\) to \(\mathbb{Z}_2^{2n}\), which is an isometry from the Lee metric on \(\mathbb{Z}_4^n\) to \(\mathbb{Z}_2^{2n}\). It is also defined coordinatewise: on \(\mathbb{Z}_4\) we have

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\gamma(0) = 00, \quad \gamma(1) = 01, \quad \gamma(2) = 11, \quad \gamma(3) = 10.
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\[
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3 \rightarrow Z_4 \rightarrow 1 \rightarrow 0
\end{array}
\end{array}
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\]

\[
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\begin{array}{c}
10 \rightarrow Z_2^2 \rightarrow 01 \rightarrow 00
\end{array}
\end{array}
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\]
A theorem and a conjecture

Conjecture

Let $C$ be a linear code over $\mathbb{Z}_4$ and $C'$ its Gray map image. Then the strength of $C'$ is one less than the minimum Lee weight of $C^\perp$. Note that the strength of $C$ is one less than the minimum Hamming weight of $C^\perp$. 

Theorem

Let $C$ be a linear code over $\mathbb{Z}_4$ and $C'$ its Gray map image. Then the strength of $C'$ is at most the minimum Lee weight of $C^\perp$ minus one.
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Moreover, if $C$ and $C'$ have strength $t$ and $t'$ respectively, then it is known that $t \leq t' \leq 2t + 1$. (This would follow from the truth of the conjecture.)
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A classification of $\mathbb{Z}_4$-codes

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Given binary codes $C_1 \leq C_2$, let $\mathcal{C}(C_1, C_2)$ be the set of all $\mathbb{Z}_4$-codes $C$ corresponding as above to the pair $C_1, C_2$. 
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**Proposition**

*If the length is $n$, and $\dim(C_i) = k_i$ for $i = 1, 2$, then $|\mathcal{C}(C_1, C_2)| = 2^{k_1(n-k_2)}$.***
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Given $C_1$ and $C_2$, what can we say about properties of the codes in $\mathcal{C}(C_1, C_2)$? 
The code $C$ has a generator matrix of the form \[
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I & X & Y \\
O & 2I & 2Z
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We can assume that $X$ is a zero-one matrix. Then $Y$ is only determined mod 2 by $C_1$ and $C_2$, so the codes in $C(C_1, C_2)$ are found by adding 0 or 2 to the elements of $Y$. 
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Since $Y$ is $k_1 \times (n - k_2)$, where $k_i = \text{dim}(C_i)$, this gives the formula for $|C(C_1, C_2)|$. 

Generator matrices
Weight enumerators

The **symmetrized weight enumerator** of a $\mathbb{Z}_4$-code $C$ is the three-variable homogeneous polynomial

$$
\sum_{c \in C} x^{n_0(c)} y^{n_2(c)} z^{n_1(c)+n_3(c)}.
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Apart from renormalisation, we obtain the weight enumerators of $C_1$ and $C_2$ by the substitutions $x \to x$, $y \to x$, $z \to y$ and $x \to x$, $y \to y$ and $z \to 0$ respectively.
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**Theorem**

The average of the symmetrized weight enumerators of the codes in \( \mathcal{C}(C_1, C_2) \) is

\[
\frac{|C_2|}{2^n} (W_{C_1}(x+y, 2z) - (x+y)^n) + W_{C_2}(x,y).
\]
Carrie Rutherford and I are currently trying to obtain further global information about this; in particular, the “variance” of the weight enumerators of codes in $C(C_1, C_2)$. Fatma Al-Kharoosi examined this situation locally, and showed that there are only a limited number of possibilities for the way that the s.w.e. changes in moving from one code in the class to a neighbouring one. A detailed example is given later.
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A detailed example is given later.
The fact that \( |C(C_1, C_2)| \) is a power of 2 is not a coincidence: the group \( C_1^* \otimes (\mathbb{Z}_2^n/C_2) \) acts on this set by translation. \((C_1^* \text{ is the dual space of } C_1\).)
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For \( C_1^* \otimes \mathbb{Z}_2^n \) acts on \( \mathcal{C} \) by the rule

\[
(f \otimes w)(c) = c + d(f(c \mod 2))w
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where \( d \) is the "doubling" map \( 0 \to 0, 1 \to 2 \) from \( \mathbb{Z}_2 \) to \( \mathbb{Z}_4 \), and the kernel of the action is \( C_1^* \otimes C_2 \).
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So if we fix a reference code in \( \mathcal{C} \) to act as origin, there is a bijection between \( \mathcal{C} \) and \( C_1^* \otimes (\mathbb{Z}_2^n / C_2) \).
Another group action

It is clear that \( \mathcal{C} \) is invariant under \( \text{Aut}(C_1) \cap \text{Aut}(C_2) \), the common automorphisms of \( C_1 \) and \( C_2 \).
Another group action

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Let $A$ be an abelian group, and $G$ a group acting on $A$. 

First cohomology
First cohomology

Let $A$ be an abelian group, and $G$ a group acting on $A$. A **derivation** is a map $d : G \to A$ satisfying $d(g_1 g_2) = d(g_1) g_2 + d(g_2)$. It is **inner** if there is an element $a \in A$ such that $d(g) = a g - a$. The derivations modulo inner derivations form a group, the first cohomology group $H^1(G, A)$, whose elements correspond bijectively to the conjugacy classes of complements of the normal subgroup $A$ in the semidirect product $A \rtimes G$. If $A$ is a vector space and $G$ a linear group, then $A : G$ is a group of affine transformations of $A$; the stabilizer of the zero vector is a complement, and a complement is conjugate to $G$ if and only if it fixes a vector.
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A very interesting case is that in which $C_1 = C_2$ is the extended Hamming code of length 8. The class $\mathcal{C}(C_1, C_2)$ includes the “octacode” whose Gray map image is the non-linear Nordstrom–Robinson code of length 16.
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A case study

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The table gives the orbit lengths of $G$ on $C$, the symmetrized weight enumerator of a code in each orbit, and the number of orbits of the subgroup $AGL(3, 2)$ (the automorphism group of the extended Hamming code). Here

$$F(x, y, z) = x^8 + 14x^4y^4 + y^8 + 16z^8 + 112xyz^4(x^2 + y^2)$$

is the weight enumerator of the octacode, and

$$E(x, y, z) = 4z^4(x - y)^4.$$
### The data

<table>
<thead>
<tr>
<th>Orbit</th>
<th>SWE</th>
<th>#perm orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>7168</td>
<td>F+5E</td>
<td>19</td>
</tr>
<tr>
<td>896</td>
<td>F+6E</td>
<td>7</td>
</tr>
<tr>
<td>21504</td>
<td>F+4E</td>
<td>24</td>
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<tr>
<td>21504</td>
<td>F+3E</td>
<td>27</td>
</tr>
<tr>
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<td>F+4E</td>
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</tr>
<tr>
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</tr>
<tr>
<td>7168</td>
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<td>8</td>
</tr>
<tr>
<td>2688</td>
<td>F+2E</td>
<td>8</td>
</tr>
<tr>
<td>128</td>
<td>F</td>
<td>3</td>
</tr>
</tbody>
</table>

The orbit of size 128 consists of octacodes. The average SWE is $F+2E$, in agreement with Theorem 6.
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The orbit of size 128 consists of octacodes.
The data

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The orbit of size 128 consists of octacodes.
The average SWE is $F + \frac{7}{2}E$, in agreement with Theorem 6.
Problems

- In the example, the symmetrized weight enumerators of the codes in $C(C_1, C_2)$ lie on a line in the space of polynomials. In general, Fatma’s work shows that they always lie on a relatively low-dimensional space. Can one calculate this dimension, in terms of $C_1$ and $C_2$?
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- Can one calculate the number of orbits of \( \mathbb{Z}_2^{n-1} : \text{Aut}(C_1) \cap \text{Aut}(C_2) \) on \( \mathcal{C}(C_1, C_2) \)? (This number is not greater than the number of orbits on \( C_1^* \otimes (\mathbb{Z}_2^n / C_2) \), and is equal if the cohomology element is zero.)
More generally . . .

Following Eric Lander’s method, we can associate a chain of $r$ codes over $\mathbb{Z}_p$ with any code over $\mathbb{Z}_{p^r}$. The $i$th code consists of words of $C$ with all entries divisible by $p^{i-1}$, read modulo $p^i$ and then “divided” by $p^{i-1}$ to give a $\mathbb{Z}_p$-code.
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Almost nothing is known about this!