

# Posets, homomorphisms, and homogeneity

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Dedicated to Jarik Nešetřil on his sixtieth birthday

## Summary

Jarik Nešetřil has made deep contributions to all three topics in the title, and we began thinking about connections between them when I spent six weeks in Prague in 2004. In this talk I want to survey the three topics and their connections. I will be reporting a theorem by my student Debbie Lockett.

- Homogeneous and generic structures
- Construction of the generic poset
- Homomorphisms and homomorphism-homogeneity
- Homomorphism-homogeneous posets

## Universality and homogeneity

A countable relational structure  $M$  belonging to a class  $\mathcal{P}$  is

- *universal* if every finite or countable structure in  $\mathcal{P}$  is embeddable in  $M$  (as induced substructure);
- *homogeneous* if every isomorphism between finite substructures of  $M$  can be extended to an automorphism of  $M$  (an isomorphism  $M \rightarrow M$ ).

The *age* of a relational structure  $M$  is the class  $\mathcal{C}$  of all finite structures embeddable in  $M$ .

## Fraïssé's Theorem

In about 1950, Fraïssé gave a necessary and sufficient condition on a class  $\mathcal{C}$  of finite structures for it to be the age of a countable homogeneous structure  $M$ .

The key part of this condition is the *amalgamation property*: two structures in  $\mathcal{C}$  with isomorphic substructures can be "glued together" so that the substructures are identified, inside a larger structure in  $\mathcal{C}$ .

Moreover, if  $\mathcal{C}$  satisfies Fraïssé's conditions, then  $M$  is unique up to isomorphism; we call it the *Fraïssé limit* of  $\mathcal{C}$ .

## Ramsey theory

There is a close connection between homogeneity and Ramsey theory.

Hubička and Nešetřil have shown that, if a countably infinite structure carries a total order and the class of its finite substructures is a Ramsey class, then the infinite structure is homogeneous.

This gives a programme for determining the Ramsey classes: first find classes satisfying the amalgamation property, and then decide whether they have the Ramsey property.

The converse is false in general, but Jarik Nešetřil recently showed that the class of finite metric spaces is a Ramsey class.

## The random graph

The class of all finite graphs is obviously a Fraïssé class. Let  $R$  be its Fraïssé limit. Then

- $R$  is the unique countable universal homogeneous graph;
- $R$  is the *countable random graph*; that is, if edges of a countable graph are chosen independently with probability  $\frac{1}{2}$ , then the resulting graph is isomorphic to  $R$  with probability 1 (Erdős and Rényi);
- $R$  is the *generic countable graph* (this is an analogue of the Erdős–Rényi theorem, with Baire category replacing measure).

### Constructions of $R$

There are a number of simple explicit constructions for  $R$ , the first of which was given by Rado.

My favourite is the following: the vertices are the primes congruent to 1 mod 4; join  $p$  to  $q$  if  $p$  is a quadratic residue mod  $q$ .

Another one (relevant to what will follow) is: Take any countable model of the Zermelo–Fraenkel axioms for set theory; join  $x$  to  $y$  if either  $x \in y$  or  $y \in x$ .

We do not need all of ZF for this; in particular, Choice is not required. The crucial axiom turns out to be Foundation.

### The generic poset

In similar fashion, the class of all finite posets is a Fraïssé class; let  $\mathbb{P}$  be its Fraïssé limit. We call  $\mathbb{P}$  the *generic poset*.

- $\mathbb{P}$  is the unique countable homogeneous universal poset;
- $\mathbb{P}$  is the generic countable poset. (It is not clear how to define the notion of “countable random poset”, but no sensible definition will give  $\mathbb{P}$ .)

Schmerl classified all the countable homogeneous posets. Apart from  $\mathbb{P}$ , there are only an infinite antichain and some trivial modifications of the totally ordered set  $\mathbb{Q}$ .

There is no known direct construction of  $\mathbb{P}$  similar to the constructions of  $R$ . I now outline a nice recursive construction by Hubička and Nešetřil.

### Set theory with an atom

Take a countable model of set theory with a single atom  $\diamond$ . Now let  $M$  be any set not containing  $\diamond$ . Put

$$M_L = \{A \in M : \diamond \notin A\},$$

$$M_R = \{B \setminus \{\diamond\} : \diamond \in B \in M\}.$$

Then neither  $M_L$  nor  $M_R$  contains  $\diamond$ .

In the other direction, given two sets  $P, Q$  whose elements don’t contain  $\diamond$ , let  $(P \mid Q) = P \cup \{B \cup \{\diamond\} : B \in Q\}$ . Then  $(P \mid Q)$  doesn’t contain  $\diamond$ .

Moreover, for any set  $M$  not containing  $\diamond$ , we have  $M = (M_L \mid M_R)$ .

Note that any set not containing  $\diamond$  can be represented in terms of sets not involving  $\diamond$  by means of the operation  $(\cdot \mid \cdot)$

For example,  $\{\emptyset, \{\diamond\}\}$  is  $(\{\emptyset\} \mid \{\emptyset\})$ .

### The generic poset

Let  $\mathcal{P}$  be the collection of the sets  $M$  not containing  $\diamond$  defined by the following recursive properties:

*Correctness:*  $M_L \cup M_R \subseteq \mathcal{P}$  and  $M_L \cap M_R = \emptyset$ ;

*Ordering:* For all  $A \in M_L$  and  $B \in M_R$ , we have

$$(\{A\} \cup A_R) \cap (\{B\} \cup B_L) \neq \emptyset.$$

*Completeness:*  $A_L \subseteq M_L$  for all  $A \in M_L$ , and  $B_R \subseteq M_R$  for all  $B \in M_R$ .

Now we put  $M \leq N$  if

$$(\{M\} \cup M_R) \cap (\{N\} \cup N_L) \neq \emptyset.$$

**Theorem 1.** *The above-defined structure is isomorphic to the generic poset  $\mathbb{P}$ .*

### Homomorphisms

A *homomorphism*  $f : M \rightarrow N$  between relational structures of the same type is a map which preserves the relations. For example, if  $M$  and  $N$  are posets with the strict order relation  $<$ , then a  $f$  is a homomorphism if and only if

$$x < y \Rightarrow f(x) < f(y).$$

As usual, a *monomorphism* is a one-to-one homomorphism, and an *isomorphism* is a bijective homomorphism whose inverse is also a homomorphism.

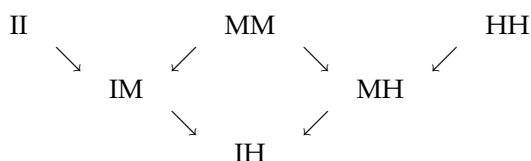
Thus, homomorphisms of the non-strict order relation in posets are not the same as homomorphisms of the strict order; but monomorphisms for the two relations are the same.

For most of this talk I will consider the strict order.

### Notions of homogeneity

We say that a relational structure  $X$  has *property HH* if every homomorphism between finite substructures of  $X$  can be extended to a homomorphism of  $X$ . Similarly,  $X$  has *property MH* if every monomorphism between finite substructures extends to a homomorphism. There are six properties of this kind that can be considered: HH, MH, IH, MM, IM, and II. (It is not reasonable to extend a map to one satisfying a stronger condition!) Note that II is equivalent to the standard notion of homogeneity defined earlier.

These properties are related as follows (strongest at the top):



### Extensions of $\mathbb{P}$

We can recognise  $\mathbb{P}$  by the property that, if  $A, B$  and  $C$  are pairwise disjoint finite subsets with the properties that  $A < B$ , no element of  $A$  is above an element of  $C$ , and no element of  $B$  is below an element of  $C$ , then there exists a point  $z$  which is above  $A$ , below  $B$ , and incomparable with  $C$ .

Extensions of  $\mathbb{P}$  (posets  $X$  with the same point set, in which  $x < y$  in  $\mathbb{P}$  implies  $x < y$  in  $X$ ) can be recognised by a similar property: if  $A$  and  $B$  are finite disjoint sets with  $A < B$ , then there exists a point  $z$  satisfying  $A < z < B$ .

Using this, it can be shown that any extension of  $\mathbb{P}$  has the properties MM and HH (and hence all the earlier properties except II).

### Properties

If an IH poset  $P$  is not an antichain, then it has the following property:

for any finite set  $Q$ , the set  $\{z : z < Q\}$  has no maximal element and  $\{z : z > Q\}$  has no minimal element.

This is easy to see in the case  $Q = \emptyset$  (so that  $P$  has no least or greatest element). In general, suppose that  $Q < z$ , and  $z < z'$ . Extend the isomorphism fixing  $Q$  and mapping  $z'$  to  $z$ ; if  $z''$  is the image of  $z$ , then  $Q < z'' < z$ .

Taking  $Q$  to be a singleton, we see that  $P$  is dense.

### X-free posets

We say that a countable poset is *X-free* if it satisfies the following:

If  $A$  and  $B$  are 2-element antichains with  $A < B$ , then there does not exist a point  $z$  with  $A < z < B$ .

Such a point  $z$  together with  $A$  and  $B$  would form the poset  $X$ .

Take a discrete tree  $T$ ; for each pair  $(x, y)$  in  $T$  such that  $y$  covers  $x$ , add a copy of the open rational interval  $(0, 1)$  between  $x$  and  $y$ ; and delete the points of  $T$ . This poset is vacuously X-free, and also has the property that for any finite  $Q$ ,  $\{z : z < Q\}$  has no maximal element and  $\{z : z > Q\}$  has no minimal element.

Any poset with these two properties can be shown to be HH and MM. This gives  $2^{\aleph_0}$  non-isomorphic HH and MM posets.

### Lockett's Theorem

**Theorem 2.** • For a countable poset which is not an antichain, the properties IM, IH, MM, MH, HH are all equivalent.

• A countable poset  $P$  has one of these properties if and only if one of the following holds:

- $P$  is an antichain;
- $P$  is the union of incomparable copies of  $\mathbb{Q}$ ;
- $P$  is an extension of the generic poset  $\mathbb{P}$ ;
- $P$  is X-free and, for any finite set  $Q$ ,  $\{z : z < Q\}$  has no maximal element and  $\{z : z > Q\}$  has no minimal element.

Thus, for posets, the earlier diagram simplifies:

$$\begin{array}{c} \text{II} \\ \downarrow \\ \text{IM} = \text{IH} = \text{MM} = \text{MH} = \text{HH} \end{array}$$

### Non-strict orders

Mono- and isomorphisms of non-strict orders are the same as for strict orders. So the classes MM and IM still coincide. However, the others are larger: a finite chain is HH, for example.

Lockett has shown that the diagram for non-strict partial order is

$$\begin{array}{c} \text{II} \\ \downarrow \\ \text{IM} = \text{MM} \\ \downarrow \\ \text{IH} = \text{MH} = \text{HH} \end{array}$$

### Graphs

For graphs, it is not true that the five classes coincide. Jarik and I showed that a countable MH graph either is an extension of the random graph  $R$  (containing it as a spanning subgraph), or has bounded claw size. Apart from disjoint unions of complete graphs (containing no  $K_{1,2}$ ), no examples with bounded claw size are known. Extensions of  $R$  are MM and HH.

The homogeneous (II) graphs were all found by Lachlan and Woodrow. They are disjoint unions of complete graphs and their complements; the Fraïssé limit of the class of  $K_n$ -free graphs ( $n \geq 3$ ) and its complement; and the random graph. We don't know what happens for IH or IM.