Synchronization and homomorphisms

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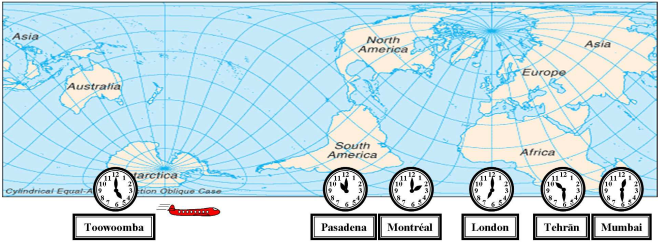
THEOREM OF THE DAY

Sims' Conjecture (1968, proved 1983) *There is a function f which, for any finite permutation group G acting primitively on a set Ω, bounds the order of a point stabilizer G_α, $\alpha \in \Omega$, as $|G_\alpha| \leq f(k)$, k being the size of a non-trivial orbit of G_α acting on Ω.*



Aviation (=imprimitive?) Local time in Toowoomba may be thought of as an independent clock face: the permutation $(1_T\ 2_T\ \ldots\ 11_T\ 12_T)$. Circling the globe permutes local time hours: $(1_T\ 1_P\ 1_{Mo}\ 1_L\ 1_{Te}\ 1_{Mu})(2_T\ 2_P\ 2_{Mo}\ 2_L\ 2_{Te}\ 2_{Mu})\ldots(12_T\ 12_P\ 12_{Mo}\ 12_L\ 12_{Te}\ 12_{Mu})$, the actual rates of change depending on your trajectory. For n cities, these two permutations generate a *wreath product* group $\mathbb{Z}_{12}$ wr $\mathbb{Z}_n$ acting on $12 \times n$ clock points. The action is *imprimitive*: clocks are always mapped to each other in entirety, never piecemeal; and the theorem does not apply: G_{1_T}, fixing time in Toowoomba, acts, regardless of n, with orbits of size $k = 12$, i.e. the other $n-1$ clocks. But $|G_{T_1}| = 12^{n-1}$, since these clocks rotate independently.

Teleportation (=primitive?) equips the traveller with a separate cycle $(1_T\ 1_P\ 1_{Mo}\ 1_L\ 1_{Te}\ 1_{Mu})$, breaking out of the clock face to give a primitive group: the symmetric or, for even n, the alternating group. (By the way, Cameron, Neumann and Teague have shown that, for almost all n, no other primitive groups exist.) Fixing 1_T now leaves all other clock points free to move, so $|G_{1_T}| = (12n-1)!$ or, for even n, $(12n-1)!/2$; and orbits have size $k = 12n - 1$. So $f(k) = k!$ works here, but to assert the existence of an f which works for *all* primitive groups is another matter!

This deep theorem, proved by Peter Cameron, Cheryl Praeger, Jan Saxl and Gary Seitz using the massive edifice of the Classification of the Finite Simple Groups, has somehow retained the name of its originator, Charles C. Sims.

Web link: designtheory.org/library/encyc/topics/permgps.pdf. The map is from `www.lib.utexas.edu/maps/world.html`.

Further reading: *Permutation Groups* by P.J. Cameron, Cambridge University Press, 1999.

Theorem of the Day is maintained by Robin Whitty at `www.theoremoftheday.org`

Thanks to Robin Whitty

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Happy birthday, Cheryl, from Robin!

List of people from Toowoomba

(from Wikipedia)

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This is part of an investigation involving, among others, João Araújo, Πeter Neumann, Jan Saxl, Csaba Schneider, Pablo Spiga, and Ben Steinberg. Nik Ruskuc, Colva Roney-Dougal, Ian Gent and Tom Kelsey have recently been involved. Some of the work also involves Cristy Kazanidis, a student of Cheryl's.

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There is far more material than can be presented here; Πeter will talk about other aspects of this topic in his workshop next week. See you there!

Notation

In this talk, X is a graph, G is a group.

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For a graph X, we use $\omega(X)$ for the clique number, $\chi(X)$ for the chromatic number, $\overline{X}$ for the complement, $\alpha(X)$ for the independence number (so that $\alpha(X) = \omega(\overline{X})$), and $\mathrm{Aut}(X)$ for the automorphism group of X.

Graph homomorphisms

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- $K_m \to X$ if and only if $\omega(X) \geq m$;
- $X \to K_m$ if and only if $\chi(X) \leq m$.

Cores

The core of X is the (unique) smallest graph Y such that $Y \equiv X$. It is an induced subgraph (indeed, a retract) of X. (This means that it is the image of an idempotent endomorphism of X.)

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- X is equal to its core if and only if all its endomorphisms are automorphisms;
- the core of X is complete if and only if $\omega(X) = \chi(X)$.

Cores and symmetry

Proposition

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Suppose that X is vertex-transitive, and let v and w be vertices of Y. Choose $g \in \mathrm{Aut}(X)$ with $vg = w$. Then $g' = \iota g \rho$ is an endomorphism of Y mapping v to w. Since Y is a core, g' is an automorphism of Y.

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Rank 3 graphs

A graph X is a rank 3 graph if its automorphism group is transitive on vertices, ordered edges and ordered non-edges; in other words, $\mathrm{Aut}(X)$ is a rank 3 permutation group. (The rank of a permutation group G on a set V is the number of G-orbits on $V \times V$.)

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This is true; the proof came from an unexpected direction: automata theory.

The cave

You are in a dungeon consisting of a number of rooms. Passages are marked with coloured arrows. Each room contains a special door; in one room, the door leads to freedom, but in all the others, to instant death. You have a schematic map of the dungeon, but you do not know where you are.

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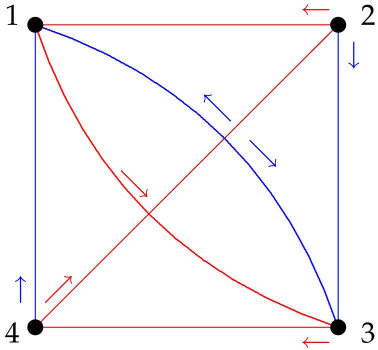

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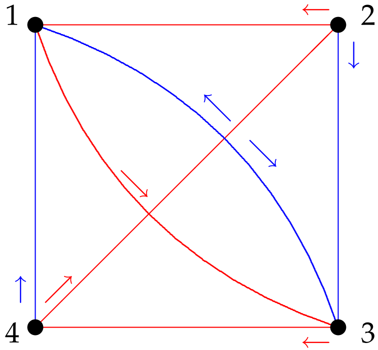

You can check that (Blue, Red, Blue, Blue) is a reset word which takes you to room 3 no matter where you start.

Automata and reset words

An automaton is an edge-coloured digraph with one edge of each colour out of each vertex. Vertices are states, colours are transitions. A reset word is a word in the colours such that following edges of these colours from any starting vertex always brings you to the same state. An automaton which possesses a reset word is called synchronizing.

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Not every finite automaton has a reset word; the Černý conjecture, states that, if a reset word exists, then there is one of length at most $(n-1)^2$, where n is the number of states (or rooms in our example).

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Algebraically, an automaton is a submonoid of the full transformation monoid T_n on $\{1, \ldots, n\}$, with a distinguished set of generators; it is synchronizing if and only if it contains a constant function.

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Note that the third condition on X is much stronger than the second. We will return to this!

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If $v \sim w$ and $f \in M$, then $vf \neq wf$ by definition. Moreover, if $vf \not\sim wf$ then $(vf)h = (wf)h$ for some h, contradicting the fact that $v \sim w$ (since $fh \in M$). So $M \leq \mathrm{End}(X)$.

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If $v \sim w$ and $f \in M$, then $vf \neq wf$ by definition. Moreover, if $vf \not\sim wf$ then $(vf)h = (wf)h$ for some h, contradicting the fact that $v \sim w$ (since $fh \in M$). So $M \leq \mathrm{End}(X)$.
Finally, if $f \in M$ has minimum rank, then the image of f carries a complete graph Y (since it cannot be made smaller by any element of M), and so Y is the core of X.

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In the forward direction, apply the preceding theorem to $M = \langle G, f \rangle$, where f is such that M contains no constant function.

In the reverse direction, if X is non-null, then no endomorphism of it is constant; and if X is not a core, then it has an endomorphism which is not an automorphism. □

Cores revisited

These considerations gave me the idea for the following theorem:

Theorem

Let X be a nonedge-transitive graph. Then either

- $\text{core}(X)$ *is complete, or*
- X *is a core.*

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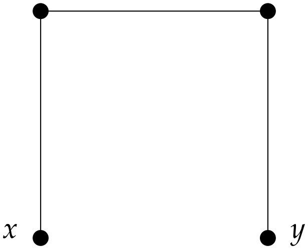
- $\mathrm{End}(X) \leq \mathrm{End}(\mathrm{hull}(X))$ *and* $\mathrm{Aut}(X) \leq \mathrm{Aut}(\mathrm{hull}(X))$*;*
- *if* $\mathrm{core}(X)$ *has m vertices then* $\mathrm{core}(\mathrm{hull}(X))$ *is the complete graph on m vertices.*

An example



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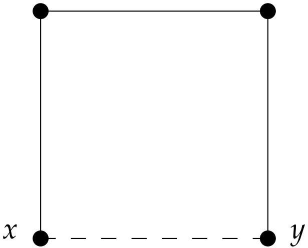

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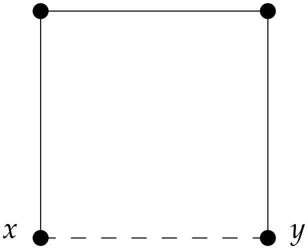

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Note the increase in symmetry: $|\operatorname{Aut}(X)| = 2$ but $|\operatorname{Aut}(\operatorname{hull}(X))| = 8$.

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Remark: For any graph X,

- $\text{hull}(X)$ is complete if and only if X is a core;
- if $\text{hull}(X) = X$ then $\text{core}(X)$ is complete (but our example shows that the converse is false).

Questions about hulls

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Problem

Given a graph X, what is the complexity of deciding:

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Given a graph X, what is the complexity of deciding:

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If the third question is hard, so are the other two.
Note that deciding if hull(X) is a complete graph is NP-complete (this is equivalent to deciding if X is a core).

Strongly regular graphs

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Godsil and Royle have some results on this question.

Homomorphisms revisited

We've seen that, if X is edge-transitive and not a core, then its core is complete, so that among its endomorphisms we have both automorphisms (of maximum rank) and proper colourings (of minimum rank).

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There may be other endomorphisms intermediate between these two extremes. For example, in a complete k-partite graph, with partite sets $B_1, \ldots, B_k$, take any map f_i from B_i to itself for $i = 1, \ldots, k$; combining these will be a homomorphism.

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Perhaps in other cases things are more restricted.

Pseudocores

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This is true for the triangular graphs $T(n) = L(K_n)$ for $n \geq 5$, for example.
Maybe this is also true for strongly regular graphs which are not complete multipartite …

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So I will state a fairly vague question here:

Problem

What can be said about core-transitive graphs?

Other classes of permutation groups

We defined the class of synchronizing permutation groups earlier, and saw that a synchronizing group is primitive. Further, such a group is basic, and so by the O'Nan–Scott theorem, it is affine (with the stabiliser of the origin a primitive linear group), simple diagonal, or almost simple.

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Not every basic group is synchronizing, however. For example, the symmetric group S_n acting on 2-sets is primitive and basic for $n \geq 5$ but is synchronizing if and only if n is odd.
There is a body of work about which such groups are synchronizing, but we do not have complete answers yet.

Other classes of permutation groups

The connection with automata leads to other interesting classes of permutation groups. For example, a permutation group G on Ω is QI if the rational permutation module $\mathbb{Q}\Omega$ is the sum of a 1-dimensional trivial module and an irreducible module. Arnold and Steinberg showed that this condition (which is stronger than synchronizing) suffices to prove the Černý conjecture for automata containing G in their transition monoid.

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In fact, a formally weaker condition which we have called spreading suffices for this implication. But as yet we have no example of a group which is spreading but not QI.

Πeter Neuman's workshop next week will give more details about this.