The profile of a relational structure

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Relational structure, age, profile

A *relational structure* is a set carrying a collection of relations with specified arities. Graphs, partial orders, circular orders, etc. are examples.

The *age* of an infinite relational structure is the class of all finite structures embeddable into it.

The *profile* is the sequence $(f_0, f_1, f_2, \ldots)$, were f_n is the number of n-element structures in the age, up to isomorphism.

Examples

- An infinite linear order
  - Age: all finite linear orders
  - Profile: $f_n = 1$ for all n
- A disjoint union of edges
  - Age: All finite unions of edges and isolated vertices
  - Profile: $f_n = \lfloor n/2 \rfloor + 1$
- An infinite path
  - Age: All finite unions of paths
  - Profile: $f_n = p(n)$ (partitions of n)
- A totally ordered set coloured with k colours, each colour class dense
  - Age: words in an alphabet of size k
  - Profile: $f_n = k^n$
- A partition into 2-sets with parts totally ordered
  - Age: Ordered partitions of finite sets into parts of size 1 or 2
  - Profile: $f_n =$ nth Fibonacci number
- A generic set with a total order and equivalence relation
  - Age: Partitioned sets
  - Profile: $f_n = B_n$ (nth Bell number)
- A universal graph
  - Age: All finite graphs
  - Profile: $f_n \sim 2^{n(n-1)/2}/n!$

Permutation groups

Let G be a permutation group on the countably infinite set Ω. Then there is a relational structure R on Ω such that

- G is contained in the automorphism group of Ω;
- if two finite substructures of R are isomorphic, then there is an element of G inducing the given isomorphism between them. This means that R is *homogeneous*, and that G is a *dense* subgroup of its automorphism group (in the topology of pointwise convergence).

So the profile of R also counts orbits of G on n-element subsets of Ω for $n = 0, 1, 2, \ldots$.

The growth of the profile

Quite a lot is known globally about the growth of a profile:

- Either $an^d \leq f_n \leq bn^d$ for some natural number d and $a, b > 0$; or f_n grows faster than a polynomial in n.
- In the latter case, $f_n \geq \exp(n^{1/2-\epsilon})$ for sufficiently large n. (These two results assume that the number of relations is finite).
- In the case of a primitive permutation group (one preserving no non-trivial equivalence relation), there is a constant $c > 1$ such that either $f_n = 1$ for all n, or $f_n \geq c^n/p(n)$ for some polynomial p.

Local conditions

Much less is known about "local" conditions relating individual values of f_n.

Theorem 1. $f_n \leq f_{n+1}$.

There are two known proofs of this theorem; one using a Ramsey-type theorem (outlined on the next slide), the other using finite combinatorics and linear algebra (see later).

A Ramsey-type theorem

Given a colouring of the n-sets with colours $c_1, \ldots, c_r$, we say that the *colour scheme* of an $(n+1)$-set S is the r-tuple $(a_1, \ldots, a_r)$, where a_i is the number of sets of colour c_i in S.

Theorem 2. *Let the n-subsets of an infinite (or sufficiently large finite) set Ω be coloured with r colours (all of which are used). Then there are at least r colour schemes of $(n+1)$-sets. In fact, there exist $(n+1)$-sets $T_1, \ldots, T_r$ so that T_i contains a set of colour c_i but none of colour c_j for $j > i$.*

The "Ramsey numbers" associated with this theorem are not known.

The age algebra

Let V_n be the complex vector space of all functions from $\binom{\Omega}{n}$ to $\mathbb{C}$ which are constant on isomorphism classes (or G-orbits). Thus, $\dim(V_n) = f_n$.

There is a multiplication defined on $A = \bigoplus_{n\geq 0} V_n$ as follows: for $f \in V_n$, $g \in V_m$, and $X \in \binom{\Omega}{m+n}$, put

$$(fg)(X) = \sum_{Y \in \binom{X}{n}} f(Y)g(X \setminus Y).$$

The multiplication is commutative and associative, and the constant function $1 \in V_0$ is the identity.

So A is a *graded algebra* with *Hilbert series* $\sum f_n x^n$.

In the fourth of our examples, A is the *shuffle algebra* on k symbols.

The structure of A

Let e be the constant function $1 \in V_1$.

Theorem 3. *The element e is not a zero-divisor in A.*

This theorem is proved by finite combinatorial arguments. It implies that multiplication by e is a monomorphism from V_n to V_{n+1}, and hence

$$f_n = \dim(V_n) \leq \dim(V_{n+1}) = f_{n+1}$$

for any n.

Two conjectures

A relational structure R is said to be *inexhaustible* if there is no point whose removal makes the age strictly smaller. In the group case, this holds if and only if G has no finite orbits.

Some time ago I conjectured the group case of the following.

Conjecture 1. *Assume that R is inexhaustible. Then*

- *A is an integral domain (that is, has no zero-divisors);*
- *e is prime in A (that is, $A/\langle e \rangle$ is an integral domain).*

The first of these conjectures has very recently been proved by Maurice Pouzet.

Local consequences

Pouzet's Theorem has the following consequence:

Theorem 4. *Assume that R is inexhaustible. Then* $f_{m+n} \geq f_m + f_n - 1$.

In outline: multiplication induces a map from the *Segre variety* (the rank 1 tensors modulo scalars) in $V_m \otimes V_n$ into V_{m+n} modulo scalars; so the dimension of V_{m+n} is at least as great as that of the Segre variety.

In a similar way, if the second part of the conjecture is true, then the profile of an inexhaustible structure would satisfy $g_{m+n} \geq g_m + g_n - 1$, where $g_n = f_{n+1} - f_n$. (Apply a similar argument to $A/\langle e \rangle$, whose nth homogeneous component is V_{n+1}/eV_n, with dimension $f_{n+1} - f_n$.)

Sketch proof

Let Ω be an set, $\mathbb{K}$ a field with characteristic zero. Let $f : \binom{\Omega}{n} \to \mathbb{K}$. The *support* of f is $\{X \in \binom{\Omega}{n} : f(X) \neq 0\}$. A set T is a *transversal* to a family $\mathcal{H}$ of sets if $T \cap H \neq \emptyset$ for all $H \in \mathcal{H}$. The *transversality* of $\mathcal{H}$ is the cardinality of the smallest transversal.

Pouzet proved:

Theorem 5. *Given $m, n \geq 0$, there exists t such that, for any Ω with $|\Omega| \geq m + n$, any field $\mathbb{K}$ of characteristic zero, and any two non-zero maps $f : \binom{\Omega}{n} \to \mathbb{K}$, $g : \binom{\Omega}{m} \to \mathbb{K}$ such that $fg = 0$, the transversality of* $\operatorname{supp}(f) \cup \operatorname{supp}(g)$ *is at most t.*

The result follows since removal of a transversal would decrease the age, which is impossible in an inexhaustible structure.

Ramsey numbers

The theorem is a Ramsey-type theorem, and one can ask for an evaluation of $\tau(m, n)$, the smallest number t for which the conclusion of the theorem is true. It is not hard to show that $\tau(1, n) = 2n$: this is the combinatorics underlying the proof that $f_n \leq f_{n+1}$.

Pouzet's proof shows that

$$7 \leq \tau(2, 2) \leq 2(R_k^2(4) + 2),$$

where $k = 5^{30}$ and $R_k^2(4)$ is the classical Ramsey number, the least p such that in any k-colouring of the edges of the complete graph on p vertices, there is a monochromatic subgraph of order 4.

This is rather a large gap – can it be reduced?

Where next?

The conjecture that, if R is inexhaustible, then e is prime in $A(R)$, remains to be proved.

A more interesting possibility involves showing that, under suitable hypotheses to be determined, if $f_1, \ldots, f_r \in V_n$ and $g_1, \ldots, g_r \in V_m$ are linearly independent, then

$$f_1 g_1 + \cdots + f_r g_r \neq 0.$$

If this were true, the dimension argument would give a much stronger lower bound for f_{m+n} in terms of f_m and f_n.

But it cannot be true in general since the earlier bound is tight in some cases!