Synchronization and permutation groups

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This is part of an investigation involving, among others, João Araújo, Πeter Neumann, Jan Saxl, Csaba Schneider, Pablo Spiga, and Ben Steinberg. Cristy Kazanidis, Nik Ruskuc, Colva Roney-Dougal, Ian Gent and Tom Kelsey have also been involved.

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See also Gordon Royle's talk at this meeting for a more combinatorial approach.

Automata

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We can represent an automaton as an edge-coloured directed graph, where the vertices are the states, and the colours are the transitions. We require that the graph should have exactly one edge of each colour *leaving* each vertex.

Synchronization

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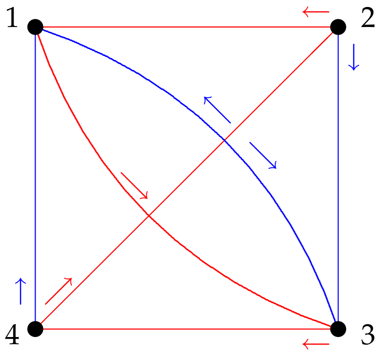
A reset word is a sequence of transitions which take the automaton from any state into a known state; in other words, the composition of the corresponding transitions is a constant mapping.

An example



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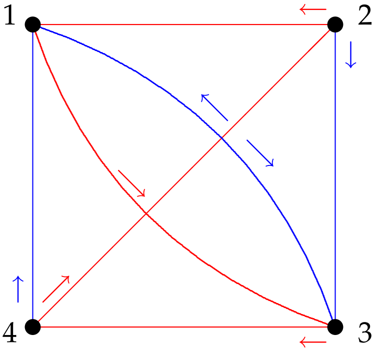

You can check that (Blue, Red, Blue, Blue) is a reset word which takes you to room 3 no matter where you start.

Applications

- Industrial robotics: pieces arrive to be assembled by a robot. The orientation is critical. You could equip the robot with vision sensors and manipulators so that it can rotate the pieces into the correct orientation. But it is much cheaper and less error-prone to regard the possible orientations of the pieces as states of an automaton on which transitions can be performed by simple machinery, and apply a reset word before the pieces arrive at the robot.

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- Bioinformatics: If a soup of DNA molecules is to perform some computation, we need the molecules to be all in a known state first. We can simultaneously apply a reset word to all of them, where the transitions are induced by some chemical or biological process.

The road-colouring problem

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This was the road-colouring conjecture until it was proved by Avraham Trahtman last year.

The Černý conjecture

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This is the Černý conjecture, and is still open. If true, the bound would be best possible.

A group-theoretic approach

At the other extreme from a synchronizing automaton is one in which all the transitions are permutations (and generate a permutation group). One approach to the Černý conjecture is to separate out this difficulty.

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Problem

Which permutation groups are synchronizing?

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Proposition

A permutation group G on Ω is non-synchronizing if and only if there is a non-trivial partition π of Ω and a subset Δ of Ω such that, for all $g \in G$, Δg is a section (of transversal) of π.

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Corollary

A synchronizing group is primitive.

For if there is a G-invariant partition π, then any section of π has the required property.

Non-synchronizing ranks

This is an attempt to measure the failure of a permutation group to be synchronizing. We define the set $M(G)$ of non-synchronizing ranks of a permutation group G to be the set of ranks of functions f on Ω for which $\langle G, f \rangle$ contains no constant function. Thus $M(G) = \emptyset$ if and only if G is synchronizing.

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- *$n - 1 \in M(G)$ if and only if G is imprimitive.*
- *$2 \in M(G)$ if and only if G has (possibly trivial) blocks B_1 and B_2 with $B_1 \subset B_2$ and $|B_2| = 2|B_1|$.*

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- *If* G *has a block of size* k, *then*

$$\{n/k, n/k+1, \ldots, n-1\} \cup \{k, 2k, \ldots, n-k\} \subseteq M(G).$$

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By contrast, we conjecture that if G is primitive then $M(G)$ is very small.

Separating groups

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Proposition

A separating group is synchronizing.

For if G is non-synchronizing, and Γ is a part of a partition π for which (π, Δ) witness the non-synchronization, then by assumption $|\Gamma \cap \Delta g| = 1$ for all $g \in G$.

Separation and synchronization

Since synchronizing groups are primitive, the obvious first step is to check primitive groups of small degree (up to a few hundred) for these properties. MAGMA and GAP contain lists of these groups. But the checking is non-trivial.

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In particular, we only know a tiny handful of permutation groups which are synchronizing but not separating; it would be interesting to find out why this property is so rare.

Some of the examples come from finite geometry (involving properties of ovoids and spreads in polar spaces), but others appear to be "sporadic".

Graph-theoretic characterisations

These properties can be detected by undirected graphs admitting the group G. The clique number $\omega(X)$ and the independence number $\alpha(X)$ are the cardinalities of the largest complete and null induced subgraphs of X; the chromatic number $\chi(X)$ is the smallest number of colours required to colour the vertices so that adjacent vertices get different colours. Clearly $\omega(X) \leq \chi(X)$, since vertices of a complete subgraph must get different colours.

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Proposition

Let G be a permutation group on Ω, with $|\Omega| = n$.

- *G is non-synchronizing if and only if there is a non-trivial G-invariant graph X for which $\omega(X) = \chi(X)$.*
- *Let G be transitive. Then G is non-separating if and only if there is a non-trivial G-invariant graph X such that $\omega(X) \cdot \alpha(X) = n$.*

Basic groups

A power structure on Ω is a hypercube with vertex set Ω, that is, a bijection between Ω aand X^n for some set X and integer $n > 1$.

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A permutation group G is non-basic if it preserves a power structure on Ω. Such a group is contained in a wreath product of smaller permutation group.

Proposition

A synchronizing group is basic.

For, if G is non-basic, then let π be the partition of X^n according to the value of the first coordinate, and Δ the diagonal set $\{(x, x, \ldots,) : x \in X\}$.

The O'Nan–Scott Theorem

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In particular, product actions of wreath products, twisted wreath products, and "compound diagonal" groups cannot be synchronizing; and an affine group in which the linear subgroup (the stabiliser of the zero vector) is imprimitive (i.e. preserves a direct sum decompositon) is not synchronizing.

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I will look at a couple of examples, to illustrate that hard problems arise!

The symmetric group acting on k-sets

Let G be the permutation group induced by S_n on the set Ω of k-subsets of $\{1, \ldots, n\}$, for $1 < k < n/2$.

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We use Baranyai's Theorem: there is a partition π of Ω into subsets each of which is a partition of $\{1, \ldots, n\}$. Take Δ to consist of the k-subsets containing the element 1.

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For $k = 2$, the following are equivalent:

- *G is synchronizing;*
- *G is separating;*
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To show the non-trivial implication, suppose that n is odd. The G-invariant graphs are $L(K_n)$ and its complement. Now $L(K_n)$ has clique number $n-1$ and independence number $\lfloor n/2 \rfloor$, so G is separating if n is odd.

The symmetric group acting on k-sets

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For $k = 3$, the following are equivalent:

- *G is synchronizing;*
- *G is separating;*
- *n is not a multiple of 3, not congruent to 1 mod 6, and not equal to 8.*

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One step in the proof depends on Teirlinck's theorem that there is a large set of Steiner triple systems if n is congruent to 1 or 3 mod 6 and $n > 7$ (a partition π of Ω into Steiner triple systems). Take Δ to consist of all 3-sets containing 1 and 2.

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For $k \geq 4$ the complete answer is not known, but synchronization and separation are not always equivalent.

A linear analogue

The linear analogue of S_n on k-sets is the linear group $\mathrm{GL}(n, q)$ acting on k-dimensional subspaces of the n-dimensional vector space, i.e. on $(k-1)$-flats of $\mathrm{PG}(n-1, q)$.

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For $k = 2$ (the action on lines of the projective space), this group is separating if and only if n is odd.
For even n, it is non-synchronizing if and only if there is a parallelism of lines in the projective space. The existence of a parallelism is known only in a few cases (when n is a power of 2, or when $n = 6$ and q is even).

Classical groups

Let G be a classical symplectic, orthogonal or unitary group, acting on the point set of the corresponding polar space (embedded in a projective space). This consists of all points which are isotropic with respect to the form. We assume that the Witt index is at least 2 (so that the poar space contains lines of the projective space).

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A maximal flat is a projective subspace of maximal dimension contained in the polar space. A spread is a partition of the polar space into maximal flats. An ovoid is a set of points meeting every maximal flat in a unique point.

Classical groups

Proposition

Let G be a classical group and $\mathcal{G}$ its associated polar space.

- *G is non-separating if and only if $\mathcal{G}$ has an ovoid.*
- *G is non-synchronizing if and only if $\mathcal{G}$ has either an ovoid and a spread, or a partition into ovoids.*

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The existence of ovoids and spreads in polar spaces is not completely resolved despite many years of study by finite geometers; this is a very hard geometric problem!

Towards the Černý conjecture

Suppose that G is a synchronizing permutation group. What further properties do we need in order that the Černý conjecture should hold for any automaton obtained by adjoining a non-permutation to a set of generators of G?

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Let f be a non-permutation. Without loss of generality, a reset word will look like

$$fg_1fg_2f\cdots fg_{r-1}f$$

for $g_1,\ldots,g_r \in G$. We need to bound r and also the expressions for $g_1,\ldots,g_r$ in terms of generators.

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Suppose that G is "large" enough that, for any set S, we can move it by an element $g_i \in G$ to a position where its inverse image under f is larger than $|S|$. Then we have $r \leq n-1$.

QI groups

Let $\mathbb{F}$ be a field of characteristic zero (or not dividing n). Then the permutation module $\mathbb{F}\Omega$ is the direct sum of a 1-dimensional submodule V_0 (the constant vectors) and an $(n-1)$-dimensional submodule V_1 (the vectors with coordinate sum zero).

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We say that G is QI if V_1 is irreducible in the case when $\mathbb{F} = \mathbb{Q}$.

Spreading groups

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Arnold and Steinberg showed that QI-groups have the property we noted earlier to approach the Černý conjecture. Later, Steinberg remarked that something less is required. The group G is not QI if and only if there exist functions v, w from Ω to the natural numbers, which are not constant and have support size greater than 1, such that $v \cdot wg$ is constant for $g \in G$.

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Then G is spreading otherwise.

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Hence we have the following hierarchy of properties of transitive permutation groups, listing in order of increasing strength:

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We currently have no example of a group which is spreading but not QI. However, all the other inclusions are strict.

Černý again

Proposition

Let G be a spreading permutation group on Ω. Then, for any map $f : \Omega \to \Omega$ which is not a permutation, there exist elements $g_1, \ldots, g_{n-2} \in G$ such that $fg_1fg_2 \cdots fg_{n-2}f$ is a constant function.

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If G is spreading and we can show that $g_1, \ldots, g_{n-2}$ have average length at most $n - 1$ in terms of a given generating set for G, then we have established an instance of the Černý conjecture.

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- Decide whether there exist parallelisms of projective spaces, and ovoids, spreads, and partitions into ovoids in classical polar spaces.