Algebraic properties of chromatic roots

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Co-authors

The problem was suggested by Sir David Wallace, director of the Isaac Newton Institute, during the programme on "Combinatorics and Statistical Mechanics" during the first half of 2008. Apart from him, others who have contributed include Vladimir Dokchitser, F. M. Dong, Graham Farr, Bill Jackson, Kerri Morgan, James Sellers, Alan Sokal, and Dave Wagner.

Chromatic roots

A proper *colouring* of a graph G is a function from the vertices of G to a set of q colours with the property that adjacent vertices receive different colours.

The *chromatic polynomial* $P_G(q)$ of G is the function whose value at the positive integer q is the number of proper colourings of G with q colours. It is a monic polynomial in q with integer coefficients, whose degree is the number of vertices of G.

A *chromatic root* is a complex number α which is a root of some chromatic polynomial.

Integer chromatic roots

An integer m is a root of $P_G(q) = 0$ if and only if the *chromatic number* of G (the smallest number of colours required for a proper colouring of G) is greater than m.

Hence *every non-negative integer is a chromatic root*. (For example, the complete graph K_{m+1} cannot be coloured with m colours.)

On the other hand, *no negative integer is a chromatic root*.

Real chromatic roots

Theorem 1.
- *There are no negative chromatic roots, none in the interval* $(0, 1)$, *and none in the interval* $(1, \frac{32}{27}]$.
- *Chromatic roots are dense in the interval* $[\frac{32}{27}, \infty)$.

The non-trivial parts of this theorem are due to Bill Jackson and Carsten Thomassen.

Complex chromatic roots

For some time it was thought that chromatic roots must have non-negative real part. This is true for graphs with fewer than ten vertices. But Alan Sokal showed:

Theorem 2. *Complex chromatic roots are dense in the complex plane.*

This is connected with the Yang–Lee theory of phase transitions.

Algebraic properties, I

We first observe that *any chromatic root is an algebraic integer*. The main question is, *which algebraic integers are chromatic roots?*

Let $G + K_n$ denote the graph obtained by adding n new vertices to G, joined to one another and to all existing vertices. Then

$$P_{G+K_n}(q) = q(q-1)\cdots(q-n+1)P_G(q-n).$$

We conclude that *if α is a chromatic root, then so is $\alpha + n$, for any natural number n.*

However, the set of chromatic roots is far from being a semiring; it is not closed under either addition or multiplication. (Consider $\alpha + \overline{\alpha}$ and $\alpha\overline{\alpha}$, where α is non-real and close to the origin.)

Algebraic properties, II

We were led to make two conjectures, as follows.

Conjecture 1 (The $\alpha + n$ conjecture). *Let α be an algebraic integer. Then there exists a natural number n such that $\alpha + n$ is a chromatic root.*

Conjecture 2 (The $n\alpha$ conjecture). *Let α be a chromatic root. Then $n\alpha$ is a chromatic root for any natural number n.*

If the $\alpha + n$ conjecture is true, we can ask, for given α, *what is the smallest n for which $\alpha + n$ is a chromatic root?*

An example

The *golden ratio* $\alpha = (\sqrt{5} - 1)/2$ is not a chromatic root, as it lies in $(0, 1)$.

Also, $\alpha + 1$ and $\alpha + 2$ are not chromatic roots since their algebraic conjugates are negative or in $(0, 1)$. However, there are graphs (e.g. the truncated icosahedron) which have chromatic roots very close to $\alpha + 2$, the so-called "golden root".

We do not know whether $\alpha + 3$ is a chromatic root or not.

However, $\alpha + 4$ is a chromatic root (the smallest such graph has eight vertices), and hence so is $\alpha + n$ for any natural number $n \geq 4$.

Quadratic roots

Theorem 3. *Let α be an integer in a quadratic number field. Then there is a natural number n such that $\alpha + n$ is a quadratic root.*

If α is irrational, then the set $\{\alpha + n : n \in \mathbb{Z}\}$ is the set of all quadratic integers with given discriminant. So it is enough to show that, for any non-square d congruent to 0 or 1 mod 4, there is a quadratic integer with discriminant d which is a chromatic root.

I will sketch the ideas behind the proof of this and partial results for higher-degree algebraic integers.

Rings of cliques

A *ring of cliques* is the graph $R(a_1, \ldots, a_n)$ whose vertex set is the union of $n + 1$ complete subgraphs of sizes $1, a_1, \ldots, a_n$, where the vertices of each clique are joined to those of the cliques immediately preceding or following it mod $n + 1$.

Theorem 4 (Read). *The chromatic polynomial of $R(a_1, \ldots, a_n)$ is a product of linear factors and the polynomial*

$$\frac{1}{q}\left(\prod_{i=1}^{n}(q - a_i) - \prod_{i=1}^{n}(-a_i)\right).$$

We call this the *interesting factor*.

Examples

- If $a_i = 1$ for all i (so that the graph is an $(n+1)$-cycle), the interesting factor is $((q-1)^n - (-1)^n)/q = (x^n - (-1)^n)/(x+1)$, where $x = q - 1$. Its roots are $2n$th roots of unity which are not nth roots (for n odd), or nth roots (for n even). In particular, if n is prime, this factor is irreducible and its Galois group is cyclic of order $n - 1$.
- If $n = 3$, the interesting factor of $R(1, 1, 5)$ is $q^2 - 7q + 11$, with roots $(7 \pm \sqrt{5})/2$. This is the eight-vertex graph promised earlier.

Quadratic integers

For $n = 3$, the interesting factor of $R(a, b, c)$ is $x^2 - (a + b + c)x + (ab + bc + ca)$. The discriminant of this quadratic is $(a + b + c)^2 - 4(ab + bc + ca)$.

It takes but a little ingenuity to show that this discriminant takes all possible values congruent to 0 or 1 mod 4.

For $n = 4$, we have a four-parameter family of cubics for the interesting factors. Are these enough to prove the $\alpha + n$ conjecture for cubic integers? (We have a long list of cubics obtained from this construction but don't seem to have hit everything!)

A higher-dimensional family

Let G be a graph whose vertex set is the union of two cliques, of sizes n and m. For $i = 1, \dots, m$, let F_i be the set of neighbours in the first clique of the ith vertex of the second. We may assume without loss of generality that the union of all the sets F_i is the whole n-clique, and that their intersection is empty.

The chromatic polynomial can be computed by inclusion-exclusion in terms of the sizes of the F_i and their intersections.

If $m = 2$, $|F_1| = a$ and $|F_2| = b$, we have a ring of cliques $R(1, a, b)$.

For $m = 3$, we get a six-parameter family of cubics as the "interesting factors". We have not been able to find suitable specialisations to prove the $\alpha + n$ conjecture using this family.

A remark on the $n\alpha$ conjecture

The only small piece of evidence is the following. If α is a root of the interesting factor of $R(a_1, \dots, a_m)$, then for any natural number n, $n\alpha$ is a root of the interesting factor of $R(na_1, \dots, na_m)$.

However, this does not generalise to arbitrary chromatic roots.

Problem 3. *Is there a graph-theoretic construction $G \mapsto F(G, n)$ such that, if α is a chromatic root of G, then $n\alpha$ is a chromatic root of $F(G, n)$?*

Galois groups

A weaker form of our conjecture (modulo the Inverse Galois Problem(!)) would assert:

Conjecture 4. *Every finite permutation group of degree n is the Galois group of an extension of $\mathbb{Q}$ generated by a chromatic root.*

This conjecture is amenable to computation. We computed the Galois groups of many of the interesting factors of rings of cliques $R(a_1, \dots, a_n)$. Note that we can assume without loss that $\gcd(a_1, \dots, a_n) = 1$.

Note also that, if n is prime, then the interesting factor is nth cyclotomic polynomial in $x = q - 1$, so that the cyclic groups of prime order all occur as Galois groups.

The next table shows what happens for small values.

Small rings of cliques

For given n, we test all non-decreasing n-tuples $(a_1, \dots, a_n)$ of positive integers with gcd 1 and $a_n \leq l$. G is the Galois group, in case the polynomial is irreducible. S_n and A_n are the symmetric and alternating groups of degree n, C_n the cyclic group of order n, V_4 the Klein group of order 4, D_n the dihedral group of order $2n$, and $\wr$ denotes the wreath product of permutation groups.

- $n = 4$, $l = 20$: 774 reducible, 3 with $G = A_3$, 7215 with $G = S_3$.
- $n = 5$, $l = 20$: 586 reducible, 6 with C_4, 5 with V_4, 360 with D_4, 6 with A_4, and 39250 times S_4. So every transitive permutation group of degree up to 4 occurs as a Galois group.
- $n = 6$, $l = 30$: 23228 reducible, one dihedral group of order 10, two Frobenius groups of order 20, three A_5, 1555851 times S_5. In this case, we are missing C_5.

More small rings

n	l	red	S_{n-1}	Other
7	15	734	113401	$C_6, S_2 \wr S_3(6)$, $S_3 \wr S_2(52), \mathrm{PGL}(2,5)(5)$
8	10	1132	22630	
9	8	152	11054	$S_4 \wr S_2(3)$
10	8	1061	18089	
11	6	29	4248	C_{10}
12	6	592	5492	
13	6	33	8415	C_{12}
14	6	884	10609	
15	6	307	15045	
16	6	1366	18813	

There are 16 transitive groups of degree 6. We have only found five of them as Galois groups.

Not overwhelming support for our conjecture!

Other families of graphs

We have done similar analysis on other families of graphs, including

- complete bipartite graphs;
- "theta-graphs" (one of these consists of p paths of length s with the endpoints identified) – these were the graphs used by Sokal to show that chromatic roots are dense in the complex plane;
- small graphs.

The results are similar but there is no time to present them here.

Further speculation

The Galois group of a "random" polynomial is typically the symmetric group of its degree.

The chromatic polynomial of a random graph cannot be irreducible, since it will have many linear factors $q - m$, for m up to the chromatic number. Bollobás showed that the chromatic number is almost surely close to $n/(2\log_2 n)$.

Wild speculation 5. *The chromatic polynomial of a random graph is almost surely a product of linear factors and one irreducible factor whose Galois group is the symmetric group of its degree.*