SEMINAR: “Stitching up $\mathbb{Z}_{pqr}$”

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Abstract

Consider the following sequence of the elements of $\mathbb{Z}_n$ where $n = 15$:

$$3\ 9\ 1\ 2\ 4\ 8\ 5\ 0\ 10\ 7\ 11\ 13\ 14\ 6\ 12\ .$$

We take the “difference” between any two consecutive entries x and y to be the value d, satisfying $0 < d < n/2$, that is congruent to either $x - y$ or $y - x \pmod n$. So the sequence's differences, in order, are

$$6\ 7\ 1\ 2\ 4\ 3\ 5\ 5\ 3\ 4\ 2\ 1\ 7\ 6\ .$$

Each of $1, 2, \ldots, 7$ occurs here exactly twice, so the original sequence is a **“terrace”** for $\mathbb{Z}_{15}$.

How was the terrace obtained? The entries after 0 are the negatives of those before 0, in reverse order. The entries before 0 are

$$3^1\quad 3^2\quad 2^0\quad 2^1\quad 2^2\quad 2^3\quad 5^1$$

modulo 15. So the terrace, up to the entry 0, is obtained by stitching together a segment containing multiples of 3 (a factor of 15), a segment containing units of $\mathbb{Z}_{15}$, and a segment containing a multiple of 5 (the other factor of 15). The terrace provides just one example of “stitch-up” constructions for terraces for $\mathbb{Z}_{pq}$ where p and q are distinct odd primes.

Producing similar constructions for terraces for $\mathbb{Z}_{pqr}$ is harder, but wow!, there are many possibilities.