# The Rado graph and the Urysohn space

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**Abstract**

Rado's graph was published in 1964; Urysohn's Polish space in 1927. There are many similarities between these two objects. These have led to new discoveries (with Anatoly Vershik) about the isometry group of Urysohn space. In this article, I describe the two objects and point out some analogies, especially regarding regular group actions.

## 1 The Rado graph

In 1964, Rado [8] defined a (simple, undirected) graph $R$ as follows. The vertices are the natural numbers (including zero), For $x < y$, the vertices $x$ and $y$ are adjacent if and only if, when $y$ is written in base 2, the $x$th digit is 1.

Rado showed that

(1) $R$ is *universal*; that is, any finite or countable graph is embeddable in $R$ (as an induced subgraph).

The key to the many remarkable properties of Rado's graph is the following property:

($*$) For any finite disjoint sets $U, V$ of $R$, there exists a vertex $z$ joined to all $u \in U$ and to no $v \in V$.

This is easily seen by taking $z = \sum_{u \in U} 2^u + 2^y$, where $y > \max(U \cup V)$.

Condition ($*$) is equivalent to the assertion

(2) If $A$ and $B$ are graphs with $|B| = |A| + 1$, then every embedding $A \to R$ can be extended to an embedding $B \to R$.

From this, the universality follows by an easy induction. Also, using ($*$) in conjunction with a back-and-forth argument, we see that any two countable graphs satisfying ($*$) are isomorphic. Moreover, starting the back-and-forth with a given finite isomorphism, we see:

(3) $R$ is *homogeneous*; that is, any isomorphism between finite induced subgraphs of $R$ can be extended to an automorphism of $R$.

Erdős and Rényi [5], to whom this analysis is due, observed that in a random countable graph (obtained by choosing edges independently with probability $1/2$ from a given countable vertex set), condition ($*$) holds with probability 1. In other words,

(4) A random countable graph is isomorphic to $R$ with probability 1.

For this reason, $R$ is often called the countable *random graph*.

One can use Baire category here in place of measure. Regarding the edge set of a graph on a fixed countable vertex set as a countable zero-one sequence, the set of graphs forms a complete metric space; and the class of graphs satisfying ($*$) (and hence isomorphic to $R$) is a residual subset (the complement of a set of first category).

Further properties of the graph $R$ can be found in [1].

# 2 Fraïssé's Theorem

In fact, much of the above analysis had been done earlier, and in much greater generality, by Fraïssé [6]. He considered *relational structures*, each consisting of a set carrying a collection of named relations. (Thus a graph is a relational structure with a single binary relation which is symmetric and irreflexive.) Induced substructures are defined in the obvious way, and embeddings are always as induced substructures. Homogeneity of relational structures is defined as for graphs: a structure is homogeneous if every isomorphism between finite induced substructures extends to an automorphism.

Fraïssé defined the *age* of a relational structure to be the class of all finite structures which are embeddable in it. He showed that a class $\mathscr{C}$ of finite relational structures is the age of a countable homogeneous structure $M$ if and only if it satisfies the following four conditions:

(a) $\mathscr{C}$ is closed under isomorphism;

(b) $\mathscr{C}$ is closed under taking induced substructures;

(c) $\mathscr{C}$ contains only countably many non-isomorphic structures;

(d) $\mathscr{C}$ has the *amalgamation property*; that is, two structures having isomorphic substructures can be "glued together" according to the isomorphism.

Moreover, if these conditions hold, then $M$ is unique up to isomorphism, and is characterised by the analogue of condition (2) of the previous section: that is, if $A, B \in \mathscr{C}$ with $|B| = |A| + 1$, then every embedding $A \to M$ can be extended to an embedding $B \to M$.

The structure $M$ is called the *Fraïssé limit* of $\mathscr{C}$. It is residual in the class of countable structures whose age is contained in $\mathscr{C}$. So at least in the sense of Baire category, we have an exact generalisation of the properties of $R$.

However, there is not always a natural sense in which it is the random structure with age contained in $\mathscr{C}$. Indeed, there may be a natural measure for which the random structure is something different! For example, the class of finite triangle-free graphs has a Fraïssé limit (the triangle-free *Henson graph* $H_3$, see [7]); but a random triangle-free graph is almost surely bipartite.

## 3 Urysohn space

A *Polish space* is a complete separable metric space. The natural class of maps for metric spaces are isometries; so a Polish space is universal if every Polish space is isometrically embeddable into it, and is homogeneous if every isometry between finite subspaces extends to an isometry of the whole space.

In a posthumous paper published in 1927, Urysohn [10] constructed the unique universal homogeneous Polish space. This paper contains ideas similar to Fraïssé's, although twenty years earlier!

The class of finite metric spaces does not satisfy Fraïssé's conditions. It does have the amalgamation property, and is obviously closed under isomorphisms and substructures; but there are too many spaces (there are uncountably many two-point spaces up to isometry). So we have to proceed a bit differently. Let $\mathscr{C}$ be the class of all finite *rational metric spaces* (those with all distances rational). This class has a Fraïssé limit, which we denote by $Q\mathbb{U}$ (the unique countable homogeneous universal rational metric space). Now the Urysohn space $\mathbb{U}$ is the completion of $Q\mathbb{U}$.

Vershik [11] has shown that $\mathbb{U}$ has properties similar to those of Fraïssé limits. It is characterised by an extension property analogous to (2) above. It is residual among metric spaces with a prescribed countable dense set, and is also the random metric space with respect to a wide class of measures.

# 4 Cyclic automorphisms

The random graph $R$ has a large and rich automorphism group $G = \mathrm{Aut}(R)$. For example, $G$ has cardinality $2^{\aleph_0}$, and $G$ is simple (a result of Truss [9]). Truss also determined all possible cycle types of elements of $G$. In particular,

(5) $R$ admits a *cyclic* automorphism (one permuting all the vertices in a single cycle).

This can be shown as follows. Let $\Gamma$ be a countable graph with a cyclic automorphism $\sigma$. Then the vertices of $\Gamma$ can be labelled $(v_i : i \in \mathbb{Z})$ such that $\sigma$ is the shift $v_i \mapsto v_{i+1}$. Let $S = \{i \in \mathbb{N} : v_i \sim v_0\}$. Then $S$ determines $\Gamma$ up to isomorphism, and $\sigma$ up to conjugacy in $\mathrm{Aut}(\Gamma)$.

Now let $S$ be a random set of positive integers (chosen independently with probability $1/2$). It is not hard to show that $\Gamma \cong R$ with probability 1. So there exists some set $S$ for which $\Gamma \cong R$, that is, $R$ admits cyclic automorphisms. Furthermore, there are $2^{\aleph_0}$ such sets, so $R$ admits $2^{\aleph_0}$ pairwise non-conjugate cyclic automorphisms.

The same conclusions can be reached (more easily) using Baire category instead of measure.

The Urysohn space is uncountable, and so cannot have a cyclic automorphism in this sense. However, the "rational Urysohn space" $Q\mathbb{U}$ has a cyclic isometry. The proof follows the above lines but is a bit more complicated in detail, and is given in [4], based on an argument from [2].

This theorem has several consequences.

- As in the case of $R$, there are uncountably many pairwise non-conjugate isometries of $Q\mathbb{U}$.

- The isometry $\sigma$ extends uniquely to Urysohn space $\mathbb{U}$, and acts with all its orbits dense. The existence of such an isometry is quite remarkable, since it is known (for example) that the only locally compact spaces having an isometry with all orbits dense are tori. Following this analogy, $(\mathbb{U}, \sigma)$ is a dynamical system which should have interesting properties.

- The closure of $\sigma$ in the full isometry group of $\mathbb{U}$ is an abelian transitive group of isometries, and so gives $\mathbb{U}$ an abelian group structure. However, different choices of $\sigma$ give rise to different abelian groups here, in particular, having different torsion subgroups.

- $\sigma$ is a bounded isometry; indeed, $d(x,x^\sigma)$ is constant for $x \in \mathbb{U}$. Hence the bounded isometries form a non-trivial proper normal subgroup of the isometry group of $\mathbb{U}$. However, the closure of this subgroup is the whole group. It should be true that $\mathbb{U}$ is simple as a topological group.

# 5 Regular automorphism groups

The results about cyclic automorphisms can be partly generalised to arbitrary countable groups.

A permutation group is *regular* if it is transitive and the stabiliser of a point is the identity. Thus every transitive abelian group is regular. A graph admits a regular action of a group $G$ if and only if it is a Cayley graph for $G$. In [2], this terminology is extended; an arbitrary object $M$ is called a *Cayley object* for $G$ if $G$ acts regularly on $M$.

For elements $u,v$ of a group $G$, let

$$S(u,v) = \{x \in G : ux^{-1} = xv^{-1}\}.$$

Note that $S(u,v) = \sqrt{uv^{-1}} \cdot v$, where $\sqrt{g}$ is the set of square roots of $g$. The relevance of this definition is that, in any $G$-invariant graph, if $x \in S(u,v)$, then $\{u,x\}$ and $\{v,x\}$ have the same character (that is, both are edges or both are non-edges).

Cameron and Johnson [3] observed that $G$ acts regularly on $R$ if and only if, for every pair $U,V$ of finite disjoint subsets of $G$, the complement of $\bigcup_{u\in U, v\in V} S(u,v)$ is infinite. In particular, if any non-identity element of $G$ has only finitely many square roots, then $G$ acts regularly on $R$. Among such groups are the cyclic group (as we have seen), any abelian group with finite 2-torsion subgroup, and the countable elementary abelian 2-group.

An example of a group for which the condition fails is the *infinite dicyclic group*

$$G = \langle a,b : b^4 = 1, b^{-1}ab = a^{-1}\rangle,$$

for which we have $G = \sqrt{b^2} \cup \sqrt{b^2} \cdot b$. In any Cayley graph for this group, for any element $x$, either $\{1,x\}$ and $\{b^2,x\}$ have the same character (both edges or both non-edges), or $\{b,x\}$ and $\{b^3,x\}$ have the same character; so no Cayley graph is isomorphic to $R$.

A *B-group* is a group $G$ with the property that any group containing the regular action of $G$ which is primitive (preserves no non-trivial equivalence relation) is doubly transitive (preserves no non-trivial binary relation at all). The B is for Burnside, who showed that finite cyclic groups of composite order are B-groups. By contrast, no infinite B-group is known. The above result shows that a wide class of countable groups are not B-groups (since $\mathrm{Aut}(R)$ is primitive but not doubly transitive).

A curious by-product is that, if some Cayley graph for $G$ is isomorphic to $R$, then almost all are (in the sense of Baire category, that is, a residual set).

The position for $Q\mathbb{U}$ is more complicated. Any countable group acting regularly on $Q\mathbb{U}$ acts regularly on $R$. Indeed, $R$ is a *reduct* of $Q\mathbb{U}$. (It is easy to construct a partition of the non-zero rationals into two disjoint sets $E$ and $N$ such that, if we call two points adjacent if and only if their distance lies in $E$, then we obtain the graph $R$. (This shows that we find no new non-B-groups among regular isometry groups of $Q\mathbb{U}$.)

However, this necessary condition is not sufficient. The countable abelian group of exponent 2 acts regularly on $Q\mathbb{U}$, but the countable abelian group of exponent 3 does not. We do not at the moment have a necessary and sufficient condition.

Note that, if we take an abelian group $A$ acting regularly on $Q\mathbb{U}$, then its closure acts regularly on $\mathbb{U}$. So $\mathbb{U}$ has the structure of a group of exponent 2.

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