The random graph has the strong small index property

Peter J. Cameron

School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK

Abstract

Hodges *et al.* showed that the countable random graph has the small index property. The stronger result of the title is deduced from this and a general theorem about permutation groups. A consequence is that the automorphism group of the random graph is not isomorphic to the automorphism group of any other countable homogeneous graph or digraph.

Key words: automorphism group, small index property, random graph

At the Fraïssé2000 meeting in Luminy, Dietrich Kuske asked for a proof that the automorphism groups of the countable universal homogeneous graph and poset are non-isomorphic. Here is such a proof.

Theorem 1 *Let R be the random graph (the unique countable homogeneous universal graph), and G any group of automorphisms of a countable graph (other than R) or digraph which is transitive on vertices, ordered edges (or arcs), and ordered non-edges. Then* $\mathrm{Aut}(R)$ *is not isomorphic to G.*

The proof depends on the following.

Theorem 2 *Let G be a permutation group on Ω having the property that, for any n-tuple $\overline{x}$ of points of Ω, all orbits outside $\overline{x}$ of the stabiliser of $\overline{x}$ in G are infinite and primitive. Let H be a subgroup of G containing the stabiliser of the n-tuple $\overline{x}$, where n is chosen minimal subject to this. Then H is contained in the setwise stabiliser of $\overline{x}$.*

PROOF. The proof is by contradiction, so assume that this does not hold. Let $\overline{x} = (x_1, \ldots, x_n)$.

Preprint submitted to Elsevier Preprint *18 June 2004*

We *claim* that there is an n-tuple in the same H-orbit as $\overline{x}$ agreeing with it in $n-1$ positions.

For by assumption, there is an element of H which does not fix the set $\{x_1, \ldots, x_n\}$; let it map x_i to y_i for $1 \leq i \leq n$, where, without loss of generality, $y_n \notin \{x_1, \ldots, x_n\}$. Now H contains the pointwise stabiliser of $x_1, \ldots, x_n, y_1, \ldots, y_{n-1}$; let U be the orbit of y_n under this group. Since U is infinite, there is an element of the stabiliser of this tuple which maps y_n to a different point y_n'. So the tuples $(y_1, \ldots, y_n)$ and $(y_1, \ldots, y_{n-1}, y_n')$ lie in the same H-orbit as $(x_1, \ldots, x_n)$. By conjugacy, there is an n-tuple $(x_1, \ldots, x_{n-1}, x_n')$ in the same orbit as $(x_1, \ldots, x_n)$. Thus the claim is proved.

Now the stabiliser of $(x_1, \ldots, x_n)$ acts primitively on the orbit of x_n. By the preceding paragraph, $H_{x_1 \ldots x_{n-1}}$ properly contains $G_{x_1 \ldots x_n}$. By primitivity, $G_{x_1 \ldots x_n}$ is a maximal subgroup of $G_{x_1 \ldots x_{n-1}}$, so we must have $H_{x_1 \ldots x_{n-1}} = G_{x_1 \ldots x_{n-1}}$, so that H contains the pointwise stabiliser of $(x_1, \ldots, x_{n-1})$. But this contradicts the minimality of n, completing the proof. $\square$

The *small index property* of a countable structure M asserts that any subgroup of $\mathrm{Aut}(M)$ of index less than $2^{\aleph_0}$ in $\mathrm{Aut}(M)$ contains the stabiliser of a finite tuple. M is said to have the *strong small index property* if any subgroup of $\mathrm{Aut}(M)$ with index less than $2^{\aleph_0}$ lies between the pointwise and setwise stabilisers of some finite tuple.

The standard example of a countable structure having the small index property but not its strong form is an equivalence relation with more than one infinite equivalence class, H being the stabiliser of such a class.

Let M be a countable structure with $\aleph_0$-categorical theory. Then all types are realised in M, and the realising sets are the orbits of $\mathrm{Aut}(M)$. We say that a type is *infinite* if it is realised by infinitely many elements, and *primitive* if $\mathrm{Aut}(M)$ acts primitively on it. If $\overline{a}$ is an n-tuple in M, a type in the theory of $(M, \overline{a})$ is *trivial* if it is realised by an element of $\overline{a}$.

Theorem 3 *Let M be a countable structure with $\aleph_0$-categorical theory. Assume that*

(a) M has the small index property;
(b) for all tuples $\overline{a}$ of M, every non-trivial type over $(M, \overline{a})$ is infinite and primitive.

Then M has the strong small index property.

PROOF. This follows immediately from the preceding result. $\square$

Theorem 4 *The random graph R has the strong small index property.*

PROOF. By Hodges *et al.* [2], R has the small index property. Moreover (see [1]), every non-trivial type in $(R, \overline{a})$ carries a copy of R on which the stabiliser of $\overline{a}$ acts homogeneously (and in particular, primitively). $\square$

Proof of Theorem 1. It follows that the primitive permutation representations of countable degree of $G = \mathrm{Aut}(R)$ are among the representations on orbits of finite sets of vertices. It is easily seen that any such representation either has permutation rank 3 (and is the action on vertices) or permutation rank at least 10 (the minimum being attained by the action on edges or on non-edges). Since an automorphism group of a digraph X satisfying the hypotheses has rank 4 (if it is not a tournament), G is not isomorphic to such a group. Moreover, G is not isomorphic to any group of automorphisms of a tournament, since G contains involutions. Finally, any orbital graph for a rank 3 action of G is isomorphic to R, so G cannot act as described on any other graph either. $\square$

References

[1] P. J. Cameron, The random graph, pp. 331–351 in *The Mathematics of Paul Erdős*, (J. Nešetřil and R. L. Graham, eds.), Springer, Berlin, 1996.

[2] W. A. Hodges, I. M. Hodkinson, D. Lascar and S. Shelah, The small index property for ω-stable ω-categorical structures and for the random graph, *J. London Math. Soc.* (2) **48** (1993), 204–218.