Ranks and signatures of adjacency matrices

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Abstract

A graph is said to be reduced if no two vertices have the same set of neighbours. It is well known that for a given natural number r, there are finitely many reduced graphs of rank r. Let $m(r)$ denote the number of vertices of the largest reduced graph of rank r. We present two constructions (the first due to Kotlov and Lovász in 1996) which show that for positive r,

$$m(r) \geq \begin{cases} 2^{(r+2)/2} - 1 & \text{if } r \text{ is even,} \\ 5 \cdot 2^{(r-3)/2} - 1 & \text{if } r \text{ is odd, } r > 1, \end{cases}$$

and we conjecture that the bounds are attained.

It is known that a number of important graph-theoretic parameters are bounded by a function of the rank of a graph and the rank is bounded by a function of the number t of negative eigenvalues. We present deeper insight in this matter. We also report on the determination of all reduced graphs with rank at most 7, and give information of the classification by rank and signature up to rank 7. This also gives (at least implicitly) an exact enumeration of all graphs with rank at most 7. We have also determined the largest reduced graphs of rank 8.

1 Introduction

A number of authors have considered the rank of the adjacency matrix of a graph, and shown that a number of graph-theoretic parameters are bounded by functions of the rank.

This paper was motivated by the question: How many graphs of order n have adjacency matrix of rank r, given n and r? This question can be answered (for labelled graphs) if we know the reduced graphs of rank r, where a graph is said to be *reduced* if no two vertices have the same set of neighbours. There are only finitely many reduced graphs of any given rank, and there is an algorithm to find them all. If the largest reduced graph with rank r has $m = m(r)$ vertices, then the number of graphs of rank r on n vertices is asymptotically cm^n for some constant c. Thus, it is important to compute m in terms of r. Kotlov and Lovász [10] gave upper and lower bounds of order $2^{r/2}$. We conjecture that the lower bound is correct, and further conjecture the complete list of all reduced graphs of rank r on $m(r)$ vertices. (The truth of this conjecture would give the precise asymptotic for the number of labelled graphs of rank r.) Our conjectures are verified for $r \leq 8$ by computation. We also give a new proof (with explicit bounds) of a theorem of Torgašev [16], according to which the rank of a graph is bounded by a function of the number of negative eigenvalues.

2 Reduced graphs

Adding isolated vertices to a graph does not change the rank of its adjacency matrix. So we may assume that our graphs have no isolated vertices.

Given any graph G, we define an equivalence relation on the vertices by setting $v \equiv w$ if v and w have the same set of neighbours. Each equivalence class is a coclique; shrinking each class to a single vertex gives a reduced graph ${}^rG$. Conversely, any graph can be constructed from a unique reduced graph by replacing the vertices by cocliques of appropriate sizes, and edges by complete bipartite graphs between the corresponding cocliques. We call this process *blowing-up*.

Various graph-theoretic properties are preserved by reducing and blowing-up. Among these is connectedness; indeed, the number of connected components which are not isolated vertices is preserved. Other parameters preserved by blowing-up include clique number and chromatic number. More important for us is the following observation. Let $\mathrm{rank}(G)$ denote the rank of the adjacency matrix of G, and $\mathrm{sign}(G)$ the signature (the pair (s,t) where s and t are the numbers of positive and negative eigenvalues respectively).

Proposition 1 $\mathrm{rank}({}^rG) = \mathrm{rank}(G)$ *and* $\mathrm{sign}({}^rG) = \mathrm{sign}(G)$.

Proof The adjacency matrix of G is obtained from that of ${}^rG$ by replacing each 0 by a block of zeros and each 1 by a block of 1s. Now the result follows by the interlacing theorem. See Chapter I of [9].

Thus, to find all graphs G with $\mathrm{rank}(G) = r$, we should find all reduced graphs with rank r and then blow them up in all possible ways.

Proposition 2 *Let H be a reduced graph on m vertices. Then the number of labelled graphs G on n vertices with ${}^rG \cong H$ is*

$$\frac{m!S(n,m)}{|\,\mathrm{Aut}(H)|},$$

where $S(m,n)$ is the Stirling number of the second kind and $\mathrm{Aut}(H)$ *is the automorphism group of H.*

Proof There are $S(n,m)$ partitions of the n vertices into m parts, and $m!/|\,\mathrm{Aut}(H)|$ ways of drawing H on the set of parts.

Note that there is an inclusion-exclusion formula for

$$S(n,m) = \frac{1}{m!}\sum_{i=1}^{m}(-1)^{m-i}\binom{m}{i}i^n.$$

So the number in the proposition is asymptotically $m^n/|\,\mathrm{Aut}(H)|$ for fixed m and large n.

3 A finiteness theorem

The main result of this section is due to Kotlov and Lovász [10]: we give the proof since we will make use of the analysis later.

Let G be a graph with adjacency matrix A of rank r and with no isolated vertices. By Theorem 8.9.1 in Godsil and Royle [7], G contains an induced subgraph H of order r whose adjacency matrix B also has rank r. Note that H must be reduced, since otherwise ${}^{r}H$ would be smaller than H and could not have rank r. Moreover, we have

$$A = \begin{pmatrix} B & BX \\ X^\top B & X^\top BX \end{pmatrix} \tag{1}$$

for some (unique) matrix X.

Theorem 3 *There are only finitely many reduced graphs whose adjacency matrix has rank r. Such a graph has at most 2^r vertices.*

Proof Observe that two columns of the matrix in Equation (1) are equal if and only if their restrictions to the first r rows are equal. So we require that each column of BX is a $(0,1)$-vector, and that none is equal to a row of B (in other words, no column of X is equal to a standard basis vector). There are at most $2^r - r$ choices for these columns (since B is invertible, each column of BX determines the corresponding column of X uniquely). So the graph has at most 2^r vertices. Now the number of such graphs is finite.

Corollary 4 *Let $G_1, \ldots, G_k$ be the reduced graphs of rank r, and let G_i have m_i vertices. Then the number of labelled graphs of rank r on n vertices is*

$$\sum_{i=1}^{k} \frac{m_i! S(n, m_i)}{|Aut(G_i)|}.$$

This is asymptotically cm^n for some constant c, where $m = \max\{m_1, \ldots, m_k\}$.

Proof This is immediate from Proposition 2. The constant c is the sum of the reciprocals of the sizes of the automorphism groups of the reduced graphs of rank r on m vertices.

We can make one simplification. Let G be a reduced graph on m vertices with no isolated vertex, and G' the graph obtained by adding an isolated vertex to G.

Then $\mathrm{Aut}(G) = \mathrm{Aut}(G')$. Now the contributions of G and G' to the sum are

$$\begin{aligned} & \frac{m!}{|\mathrm{Aut}(G)|}S(n,m) + \frac{(m+1)!}{|\mathrm{Aut}(G)|}S(n,m+1) \\ = & \frac{m!}{|\mathrm{Aut}(G)|}(S(n,m)+(m+1)S(n,m+1)) \\ = & \frac{m!}{|\mathrm{Aut}(G)|}S(n+1,m+1). \end{aligned}$$

So we only need to consider reduced graphs without isolated vertices, if we replace $S(n,m_i)$ by $S(n+1,m_i+1)$ in the sum.

Problem Calculate the function m, where $m(r)$ is equal to the order of the largest reduced graph with rank r. The value of this function is the exponential constant in the asymptotic formula for the number of labelled graphs with rank r.

Algorithm We now present an algorithm to find the reduced graphs of rank r, with no isolated vertices, for given r.

Step 1 Find all the graphs on r vertices which have rank r.

Step 2 Let H be such a graph, with adjacency matrix B. For each zero-one column vector v which is not a column of B, find the unique vector x such that $Bx = v$. Test whether $x^\top Bx = 0$, and keep the vector x if so. Let $(x_1,\dots,x_k)$ be the list of such vectors. (These correspond to all the ways of adding one vertex to H without increasing the rank.)

Step 3 Form an auxiliary graph on the vertex set $\{1,\dots,k\}$ as follows. For each two distinct indices i and j, put an edge from i to j if and only if $x_i^\top Bx_j \in \{0,1\}$. Find all cliques in this graph.

For each clique $C = \{i_1,\dots,i_t\}$, let X be the matrix with columns $x_{i_1},\dots,x_{i_t}$, and let

$$A = \begin{pmatrix} B & BX \\ X^\top B & X^\top BX \end{pmatrix}$$

as in Equation (1). Then A is the adjacency matrix of a graph with rank r.

Step 4 If we want the list of unlabelled graphs, make the collection of all such matrices, for all choices of H. Test for isomorphism and return a list of isomorphism types.

The algorithm works because, by Equation (1), the adjacency matrix must have the required form; and the entries of $X^\top BX$ must be zero or one, with zero on the diagonal.

Note that we can improve the algorithm as follows. A reduced graph can have at most one isolated vertex. So, if we determine all reduced graphs with no isolated vertices, then we obtain the rest by adding one isolated vertex to each of them. This can be done by modifying Step 2 of the algorithm to exclude the case $v = 0$ as well as the case that v is a column of B. In the calculations below we always use this modified version.

4 Two constructions

Kotlov and Lovász [10] proved that $m(r) \leq c2^{r/2}$ for some explicit constant c (which they did not calculate). They also gave, with a sketch of a proof, a construction showing that $m(r+2) \geq 2m(r)+1$:

Construction A Let G be a reduced graph with $n = m(r)$ vertices and rank r. (Note that G has an isolated vertex v.) Blow up G by doubling each vertex w into two vertices w_0 and w_1; then add one vertex a joined to w_0 for all w. Clearly the resulting graph G^* is reduced and has $2n+1$ vertices. Now it can also be described as follows: blow up $G-v$ by doubling each vertex; add a vertex a joined to one of each pair, a vertex v_0 joined only to a, and an isolated vertex v_1. After the doubling, the rank is still r; then by Theorem 2 of [3], G^* has rank $r+2$. Moreover, G^* has a unique vertex of degree 1; its neighbour is fixed by all automorphisms, so that $\mathrm{Aut}(G^*) = \mathrm{Aut}(G)$.

We also give another construction which builds a reduced graph of rank $r+2$ on $2n+1$ vertices from one of rank r on n vertices. It is more specialized, but gives more information.

Construction B Let G be a reduced regular graph on $2m$ vertices with degree m and rank r with adjacency matrix A_m, having non-zero eigenvalues $m, \lambda_2, \ldots, \lambda_r$. Let G^+ be obtained from G as follows: Blow up each vertex w of G into two vertices w_1 and w_2; then add two new vertices a_1 and a_2, where a_i is joined to

w_i for all vertices w of G, and a_1 and a_2 are joined to each other. Then G^+ is a reduced regular graph of degree $2m+1$ on $2(2m+1)$ vertices, with rank $r+2$ and non-zero eigenvalues $2m+1, 2\lambda_2, \ldots, 2\lambda_r, \theta_1, \theta_2$, where θ_1 and θ_2 are the roots of the equation $x^2+x-2m=0$.

The facts that G^+ is reduced and regular are straightforward. Also, if $m \neq 1$, then $|\operatorname{Aut}(G^+)| = 2|\operatorname{Aut}(G)|$. For clearly there is an automorphism interchanging the vertices x_1 and x_2 for all x. Also the vertices a_1 and a_2 have complementary neighbour sets and have the property that, if they are deleted from the graph, then each equivalence class of vertices has cardinality 2; if $m > 1$ then no other pair of vertices has these properties. (For $m = 1$, we have $G = K_2$ and G^+ is the triangular prism $K_2 \,\square\, K_3$, so that $|\operatorname{Aut}(G)| = 2$ and $|\operatorname{Aut}(G^+)| = 12$.)

We compute the rank of $A(G^*)$ and $A(G^+)$, and find in each case a basis for the null space.

It is straightforward to see that there are six $\{-1,0,1\}$-eigenvectors for the triangular prism $K_2 \,\square\, K_3$, as follows

Eigenvalues	Transposed eigenvectors
3	$[1,1,1,1,1,1]$
-2	$[1,0,-1,-1,0,1]$
-2	$[1,-1,0,-1,1,0]$
1	$[-1,-1,-1,1,1,1]$
0	$[-1,0,1,-1,0,1]$
0	$[-1,1,0,-1,1,0]$

The adjacency matrices of G^+ and G^* are

$$A_m^+ = \begin{pmatrix} A_m & A_m & 0 & j \\ A_m & A_m & j & 0 \\ 0 & j^\top & 0 & 1 \\ j^\top & 0 & 1 & 0 \end{pmatrix}, A_m^* = \begin{pmatrix} A_m & A_m & j & 0 \\ A_m & A_m & 0 & 0 \\ j^\top & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where j is the all-1 column vector. If X is an eigenvector of A_m orthogonal to j, with eigenvalue λ, then $[X^\top, X^\top, 0, 0]^\top$ and $[X^\top, -X^\top, 0, 0]^\top$ are eigenvectors of A_m^+ with eigenvalues 2λ and 0 respectively. Any other eigenvalue of A_m^+ is an eigenvalue of the matrix

$$B_m = \begin{pmatrix} m & m & 0 & 1 \\ m & m & 1 & 0 \\ 0 & 2m & 0 & 1 \\ 2m & 0 & 1 & 0 \end{pmatrix},$$

since if $[a,b,c,d]^\top$ is an eigenvector of B_m with eigenvalue λ, then the vector $[aj^\top, bj^\top, c, d]^\top$, where j is of size $2m$, is an eigenvector of A_m^+ with eigenvalue λ. It is readily checked that the characteristic polynomial of B_m is $(x-2m-1)x(x^2+x-2m)$. Note that if θ is a root of polynomial $f_m(x) = x^2+x-2m$, then $-1-\theta$ is also a root. It is easy to see that $[1,-1,-\theta,\theta]^\top$ and $[1,-1,1+\theta,-1-\theta]^\top$ are two eigenvectors of B_m with eigenvalues θ and $1-\theta$, respectively. Thus $Z_1 = [j^\top, -j^\top, -\theta, \theta]^\top$ and $Z_2 = [j^\top, -j^\top, 1+\theta, -1-\theta]^\top$ are eigenvectors of A_m^+ with eigenvalues θ and $-1-\theta$, respectively.

We show that the nullity of A_m^+ is $4m-r$. Suppose that W is the null space of A_m. Then for any X and Y in W, the vector $[X,Y,0,0]$ is a vector in the null space A_m^+. But the number of independent vectors of this kind is $2\dim W = 4m-2r$. Since $\operatorname{rank} A_m = r$, A_m has exactly r non-zero eigenvalues such as $\lambda_1,\ldots,\lambda_r$, with eigenvectors $X_1,\ldots,X_r$. For any i, $1 \le i \le r$, $[X_i^\top, -X_i^\top, 0, 0]^\top$ is contained in the null space of A_m^+ and these vectors are independent of the previous vectors. Thus the null space of A_m^+ has dimension $4m-r$ (since by the above argument A_m^+ has at least $r+2 = (r-1)+3$ non-zero eigenvalues). But we know that two vectors $T_1 = [Z_1^\top, -Z_1^\top, 0, 0]^\top$ and $T_2 = [Z_2^\top, -Z_2^\top, 0, 0]^\top$ are contained in the null space of A_{m+1}^+. Now using these vectors we construct two independent $\{-1,0,1\}$-vectors in the null space of A_{m+1}^+. Indeed two vectors $T_1 - T_2$ and $T_1 + (\theta/(1-\theta))T_2$ are in the null space of A_{m+1}^+. Now we want to give a $\{-1,0,1\}$-basis for the null space A_m^+. We proceed by induction on m. Clearly $4m-r$ independent vectors that we obtained using the null space of A_m are $\{-1,0,1\}$-vectors in the null space A_m^+. Also if X is a $\{-1,0,1\}$-vector corresponding to a non-zero eigenvalue of A_m, then $[X^\top, -X^\top, 0, 0]^\top$ is a $\{-1,0,1\}$-vector in the null space of A_m^+. Now if Z_1 and Z_2 are two non-$\{-1,0,1\}$-vectors corresponding to two non-zero eigenvalues of A_m which obtained using the roots of $f_i(x)$ for some $i < m$, then as we saw before we can obtain two $\{-1,0,1\}$-vectors in the null space A_m^+.

Also it is seen that any eigenvalue of A is an eigenvalue for A_m^* and other eigenvalue of A_m^* is an eigenvalue of the matrix

$$C_m = \begin{pmatrix} m & m & 1 & 0 \\ m & m & 0 & 0 \\ 2m & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of C_m is $x(x^3 - 2mx^2 - (2m+1)x + 2m^2 + 2m)$.

Note that, if $G+v$ has n vertices and rank r, then $G^+ + v$ has $2n+1$ vertices and rank $r+2$, just as in Construction A.

Now we propose two questions about the null space and the image space of graphs.

Question 1 For which graphs there is a $\{-1,0,1\}$-basis for the null space?

Question 2 For which graphs there is at least a non-zero $\{0,1\}$-vector in the row space which does not occur as a row?

If the answer to Question 2 is affirmative for all graphs with at least one edge, then by considering the related vector x in Theorem 3 of [3], we see that $m(r)$ is an increasing function. We cannot prove this, but we conjecture the precise value of $m(r)$ (see below).

Proposition 5 *Based on the constructions A and B we have*

$$m(r) \geq \begin{cases} 2^{(r+2)/2} - 1 & \text{if } r \text{ is even,} \\ 5 \cdot 2^{(r-3)/2} - 1 & \text{if } r \text{ is odd, } r > 1. \end{cases}$$

For the values of n equal to the right hand side of the above inequality, there exist at least $\lfloor r/2 \rfloor$ *non-isomorphic reduced graphs of rank r on n vertices.*

Proof Apply either of the constructions, starting with $G = K_2$ if $r = 2$ and $G = K_2 \square K_4$ if $r = 5$. The lower bound for the number of graphs is proved inductively; we can apply Construction A to all the graphs of rank r and Construction B to the regular graph.

The case $r = 3$ is trivial; the extremal graph is K_3 with an isolated vertex.

Conjecture The bound on $m(r)$ in Proposition 5 is attained for all $r \geq 2$, and the extremal graphs are just the ones given by Constructions A and B for $r \neq 2,3,5$.

If this conjecture is true, then the $\lfloor r/2 \rfloor$ reduced graphs with rank r on $m(r)$ vertices have automorphism groups of orders 2, 12, 24, $\ldots$, $3 \cdot 2^{r/2}$ if r is even, and orders 6, 48, 96, $\ldots$, $3 \cdot 2^{(r+3)/2}$ if r is odd. Summing the reciprocals of these numbers would give the constant c in the asymptotic formula $cm(r)^n$ for the number of labelled graphs with rank r on n vertices.

5 Parameters bounded by a function of the rank

We now discuss domination. The *domination number* $\gamma(G)$ of G is the smallest number of vertices with the property that the union of their closed neighbourhoods contains all vertices; the *total domination number* $\gamma_t(G)$ is similarly defined using open neighbourhoods. (The open neighbourhood $N_G(v)$ is the set of vertices joined to v; the closed neighbourhood is $N_G[v] = \{v\} \cup N_G(v)$.)

Proposition 6 *Each of the following graph parameters is bounded by a function of the rank of a graph G:*

- *number of connected components (other than isolated vertices);*
- *clique number;*
- *chromatic number;*
- *smallest number of factors in an edge partition into complete k-partite graphs, for fixed k;*
- *domination number $\gamma(G)$ and total domination number $\gamma_t(G)$ (if there are no isolated vertices);*
- *diameter (if the graph is connected);*
- *the order of the largest composition factor of the group* $\mathrm{Aut}(G)$ *which is not an alternating group.*

Proof Each of the first four parameters is unaffected by blowing up or adding isolated vertices. For domination number, see Proposition 9 below. It is well known that a graph of diameter d has more than d distinct eigenvalues, since $I, A, \ldots, A^d$ are linearly independent; so its rank is at least d. (Alternatively, blowing up doesn't change the diameter except from 1 to 2.) Finally, the automorphism group of G has a normal subgroup N (fixing all the equivalence classes of the relation $\equiv$) which is a direct product of symmetric groups on the equivalence classes; the composition factors of N are thus alternating groups and cyclic groups of order 2. The factor group G/N is a subgroup of $\mathrm{Aut}({}^rG)$, and so its composition factors have bounded order.

For each of the parameters in the theorem, we can now pose the problem of finding the best bound for that parameter in terms of the rank. We give a few examples. Let G be a graph of rank r.

- The number of connected components (other than isolated vertices) is at most $\lfloor r/2 \rfloor$; equality holds if and only if at most one such component is complete tripartite and all the rest are complete bipartite.

- The clique number of G is at most r, with equality if and only if G is complete r-partite (possibly with isolated vertices). The proof is given below.

- It was conjectured by van Nuffelen [13] that the chromatic number of a graph does not exceed its rank. This was disproved by Alon and Seymour [1]; and Raz and Spieker [14] showed that in fact the chromatic number is not bounded by any polynomial function of r.

- If G has no isolated vertices, then $\gamma_t(G) \le r$; equality is realised by a disjoint union of complete bipartite graphs, and only by these (see below).

- The path on m vertices, for m odd, has diameter and rank $m-1$, attaining the bound in the proposition.

- It is not easy to find the best bound in terms of the rank for the order of the largest non-alternating composition factor. For $n > 2$, the group $\mathrm{PSL}(n,q)$ is a composition factor of the automorphism group of a graph on $2(q^n-1)/(q-1)$ vertices having full rank (the incidence graph of points and hyperplanes in the projective space). The order of this group is roughly q^{n^2}, which is greater than r^n; so there is no bound which is polynomial in r.

The relation between clique number and rank is as follows. Note that a complete graph has full rank.

Proposition 7 *The only reduced graph of rank r with no isolated vertices which contains K_r is K_r itself.*

Proof We have $B = J - I$, from which we find that $B^{-1} = J/(r-1) - I$. Take a $(0,1)$-vector v, in which the sum of the entries is s. Then $x = B^{-1}v = sj/(r-1) - v$, where j is the all-1 vector. So $x^\top Bx = x^\top v = s^2/(r-1) - s$. So we require $s = 0$ or $s = r-1$. But if $s = 0$, then there is an isolated vertex; and if $s = r-1$, then v is one of the columns of B, and the graph is not reduced.

Corollary 8 *A graph G of rank r has clique number at most r; equality holds if and only if G is complete r-partite (possibly with some isolated vertices).*

If G has no isolated vertices, then

$$\gamma(G) \le \gamma_t(G) = \gamma_t({}^rG),$$

since a minimal total dominating set contains at most one point from each equivalence class. Moreover, a dominating set must either include every vertex in a given equivalence class, or at least one vertex dominating that class. So, if the equivalence classes are sufficiently large (bigger than $\gamma_t({}^rG)$ will suffice), then a minimum dominating set contains at most one vertex from each equivalence class, and so is a total dominating set.

Proposition 9 *Let G be a graph with no isolated vertices, having rank r. Then*

$$\gamma(G) \le \gamma_t(G) \le r.$$

Moreover, $\gamma(G) = r$ if and only if each component of G is a complete bipartite graph $K_{k,l}$ with $k,l \ge 2$; and $\gamma_t(G) = r$ if and only if each component of G is complete bipartite.

Proof After re-ordering if necessary, the adjacency matrix of G has the form

$$A = \begin{pmatrix} B & BX \\ X^\top B & X^\top BX \end{pmatrix},$$

where B is an $r \times r$ matrix and $\mathrm{rank}(B) = r$. Since there are no isolated vertices, BX has no zero columns, so every vertex outside the set consisting of the first r vertices has a neighbour among the first r vertices, which thus form a total dominating set.

Suppose that this dominating set is minimal. Then, for any $i \le r$, there exists $j > r$ such that v_j is joined to v_i and to no other vertex among $v_1, \ldots, v_r$. Choose one such vertex v_j for each v_i with $i \le r$, and re-order the vertices so that $j = i + r$ for $i = 1, \ldots, r$. Let Y be the submatrix consisting of the first r columns of X. Then $BY = I$, so $Y = B^{-1} = Y^\top$, and $Y^\top BY = B^{-1}$. It is easy to check that, if a graph H has the property that $A(H)^{-1}$ is the adjacency matrix of a graph, then H is a matching and $A(H)^{-1} = A(H)$. Applying this to the subgraph H on $\{v_1, \ldots, v_r\}$, we see that this subgraph is a matching, and that v_{i+r} is joined to v_{j+r} if and only if v_i is joined to v_j. So the induced subgraph on $\{v_1, \ldots, v_{2r}\}$ is a disjoint union of 4-cycles.

Suppose that w is an arbitrary vertex joined to more than one vertex v_i with $i \le r$, say to $v_1, \ldots, v_k$. It is easy to see that $v_1, \ldots, v_k$ are pairwise non-adjacent.

Now replace the neighbour of v_k in H by w to obtain another graph of rank r on r vertices which is not a matching, and hence not a minimal dominating set. Thus no such vertex exists. It follows that G is obtained by blowing up H, as claimed. The converse is straightforward.

Now suppose that G satisfies $\gamma_t(G) = r$; without loss of generality, G is reduced. Our remarks before the proposition show that it is possible to blow up G to a graph H with $\gamma(H) = r$. By the previous part of the proof, H (and hence G) is obtained by blowing up a matching. Again the converse is clear.

6 Positive and negative eigenvalues

Suppose that G is a graph of rank r, and write its adjacency matrix in the form

$$\begin{pmatrix} B & BX \\ X^\top B & X^\top BX \end{pmatrix},$$

as in the proof of the finiteness theorem. The columns $x_1,\ldots,x_n$ of $(I\,X)$ are vectors in $\mathbb{R}^r$, which are singular with respect to the non-degenerate quadratic form $Q(x) = x^\top Bx$, since $X^\top BX$ has zeros on the diagonal.

The set of columns corresponding to any coclique in G has all inner products 0 with respect to the form Q, and so is contained in a totally singular subspace S.

The quadratic form Q is indefinite with signature $(s, r-s)$, where s is the number of positive eigenvalues of B. (Note that sometimes the signature is defined to be the single integer $s-(r-s)$.) The largest totally singular subspace of this form has dimension $\min\{s, r-s\} \le \lfloor r/2 \rfloor$. The columns of $(B\,BX)$ corresponding to a coclique are $(0,1)$-vectors lying in the image of S under B. Now a set of $(0,1)$-vectors in a space of dimension s has cardinality at most 2^s, as can be seen by an argument like that of the finiteness theorem. So we have:

Proposition 10 *If G is a reduced graph of rank r and has s positive eigenvalues, then*

$$\alpha(G) \le 2^{\min\{s,r-s\}} \le 2^{\lfloor r/2 \rfloor}. \qquad \square$$

The bound in terms of r is attained for all r. This can be seen by observing that it is true for $r = 2$ and $r = 3$, and that if G^+ is the graph obtained from G by Construction A, then $\alpha(G^+) = 2\alpha(G)$ (and of course $\mathrm{rank}(G^+) = \mathrm{rank}(G) + 2$). A simple explicit construction goes as follows. Let L be the bipartite incidence graph of elements and subsets of an m-set. Then the adjacency matrix of L has the form

$$\begin{pmatrix} O & N \\ N^\top & O \end{pmatrix},$$

where N is $m \times 2^m$ and has rank m. Clearly L has rank $2m$ and contains a coclique of size 2^m. A similar argument shows that if we add an arbitrary number of edges to the the smaller part of L, for the resultant graph also the equality holds.

Proposition 11 *Let G be a reduced graph of rank r and $\alpha(G) = 2^{\lfloor r/2 \rfloor}$. Then r is even and G is the bipartite incidence graph of elements and subsets of an $r/2$-set, as well as some arbitrary edges added to the part of size $r/2$. Moreover every such graph is a reduced graph of rank r and $\alpha(G) = 2^{r/2}$.*

Proof Let

$$A = \begin{pmatrix} O & B \\ B^\top & C \end{pmatrix},$$

be the adjacency matrix of G and O is a zero matrix of size $2^{\lfloor r/2 \rfloor}$. Clearly if $\mathrm{rank}(B) = s$, then we have $\mathrm{rank}(A) \geq 2s$. It implies that $\mathrm{rank}(B) \leq \lfloor r/2 \rfloor$. Now if $\mathrm{rank}(B) = s < \lfloor r/2 \rfloor$, then the number of distinct $(0,1)$-vectors lying in the row space of B is at most 2^s. Since A is reduced all rows of B are distinct, a contradiction. Thus we conclude that $\mathrm{rank}(B) = \lfloor r/2 \rfloor$. Without loss of generality assume that the first $\lfloor r/2 \rfloor$ columns of B are independent. If B' is the matrix of order $2^{\lfloor r/2 \rfloor} \times \lfloor r/2 \rfloor$ obtaining by the submatrix of B induced on the first $\lfloor r/2 \rfloor$ columns of B, then B' has no repeated rows, because any column of B is a linear combination of the columns of B' and G is a reduced graph. First assume that r is even. Then

$$\mathrm{rank}(\begin{pmatrix} B \\ C \end{pmatrix}) = \mathrm{rank}(B) = r/2.$$

We know that any $(0,1)$-vector of size $r/2$ appears as a row of B' once. We claim that B has exactly $r/2$ columns. Indeed if B has some columns , say v, not in B', since the identity matrix of size $r/2$ is a submatrix of B and v is a $(0,1)$-vector, we conclude that this column should be a linear combination of columns of B' with coefficients 0 and 1. Now consider all rows of B' with exactly two 1s. If in the linear combination at least two coefficients are 1, then one component of v is at least 2, a contradiction. Thus v is a column of B'. Since the first $r/2$ columns of the matrix

$$\begin{pmatrix} B \\ C \end{pmatrix}$$

is a basis for its column space, we find that A has two equal columns, a contradiction. Thus our graph is L as well as some additional edges in its small part.

Now let r be odd. We want to show that $\alpha(G) < 2^{\lfloor r/2 \rfloor}$. Let H be a maximal subgraph of G containing $\alpha(G)$ independent vertices and all vertices corresponding to B' such that $\operatorname{rank}(H) = 2\lfloor r/2 \rfloor$. We have $G \neq H$. Add a new vertex $u \in G \setminus H$ to H and name this graph by H'. We show that $\operatorname{rank}(H') = 2\lfloor r/2 \rfloor$ or $\operatorname{rank}(H') = 2\lfloor r/2 \rfloor + 2$, which is a contradiction. If H'' is the submatrix of H' formed by deleting the last row of H' and u' is the last column of H'', then we have two cases:

1) u' is not a combination of other columns of H''. In this case by Theorem 3 of [3], the result follows.

2) u' is a linear combination of other columns of H''. In this case as we saw before u' is equal to the one of the columns correspondence to B'. If we extend this column to the corresponding column in H, then we show that this column is the same as u. To see this it is enough to check that the last entries of these columns are the same. But since the adjacency matrix corresponding to the vertices of H' not containing $\alpha(G)$ vertices of H' is a symmetric matrix, we conclude that the last entries of mentioned vectors are 0. Thus H' is a non-reduced graph and so $\operatorname{rank}(H') = 2\lfloor r/2 \rfloor$.

The bound of Proposition 10 has the following consequence, first proved by Torgašev [16]:

Theorem 12 *The rank of a graph is bounded by a function of the number of negative eigenvalues.*

Proof It is enough to show that, if G is a reduced graph with no isolated vertices, then the number n of vertices of G is bounded by a function of the number t of negative eigenvalues of t. Since K_{t+2} has $t+1$ negative eigenvalues, by the interlacing theorem (see [9, Chapter I]), we must have $\omega(G) \leq t+1$. Also, by Proposition 10, since G is reduced, we have $\alpha(G) \leq 2^t - 1$ (we subtract 1 because there is no isolated vertex). Now Ramsey's Theorem implies that $n \leq R(t+2, 2^t) - 1$, and we are done.

Remarks 1. The complete graphs show that there is no bound for the rank in terms of the number of positive eigenvalues.

2. Let $f(t)$ be the largest rank of a graph with t negative eigenvalues. The proof of the Theorem 12 gives $f(t) \leq R(t+2, 2^t) - 1$. For $t = 1$, this is best possible: $R(3,2) - 1 = 2$, and K_2 satisfies the conditions. For larger t, we can assume that the graph is reduced and not K_{t+1}, so that it contains no K_{t+1}, and we have $f(t) \leq R(t+1, 2^t) - 1$. This bound is still probably much too large. Torgašev [16] showed that $f(2) = 5$ and $f(3) = 9$.

3. In the other direction, since $\overline{L(K_n)}$ has rank $n(n-1)/2$ and $n-1$ negative eigenvalues for $n \geq 5$, we have $f(t) \geq t(t+1)/2$ for $t \geq 4$.

4. This theorem and Proposition 6 show that each of the parameters in that Proposition is bounded by a function of the number of negative eigenvalues. In each case, we can pose the problem of determining the best possible bound.

7 Small rank

A reduced graph of rank 0 is an isolated vertex, and there are no graphs of rank 1 (since the sum of the eigenvalues of the adjacency matrix is zero). For $r = 2$ and $r = 3$, it is easy to see that the only graph of rank r on r vertices is the complete graph K_r. By Proposition 7, the only reduced graphs of rank r are K_r with or without one isolated vertex, so that $m(r) = r + 1$ in these cases.

We have computed all reduced graphs of rank r for $r \leq 7$; for $r = 8$, we have computed just those of maximum order. The results are summarized in Table 1.

r	2	3	4	5	6	7	8
Max no. vertices	3	4	7	9	15	19	31
No. of graphs	2	2	18	50	3308	46892	??
No. of max. order	1	1	2	2	3	3	4

Table 1: Reduced graphs with given rank

These computations were performed in GAP [6] at the Scientific Computing Center at IPM and in London. The GAP package `GRAPE` [15] was used for the clique finding and isomorphism testing (via its interface with `nauty` [11]). We also made use of Brendan McKay's list [12] of graphs on eight vertices in order to save time in Step 1 of the algorithm in this case.

Table 2 gives data on the reduced graphs with no isolated vertex, by rank r, number of negative eigenvalues t, and number of vertices n. The negative eigenvalues were counted by Sturm's Theorem (see [2, p.198]) so that exact rational arithmetic could be used.

$r = 4$

$t \setminus n$	4	5	6	Total
2	3	3	2	8
3	1	0	0	1
Total	4	3	2	9

$r = 5$

$t \setminus n$	5	6	7	8	Total
2	1	0	0	0	1
3	7	7	7	2	23
4	1	0	0	0	1
Total	9	7	7	2	25

$r = 6$

$t \setminus n$	6	7	8	9	10	11	12	13	14	Total
3	39	142	315	428	371	204	70	15	3	1587
4	17	19	19	9	2	0	0	0	0	66
5	1	0	0	0	0	0	0	0	0	1
Total	57	161	334	437	373	204	70	15	3	1654

$r = 7$

$t \setminus n$	7	8	9	10	11	12	13	14	15	16	17	18	Total
3	25	50	61	30	11	2	0	0	0	0	0	0	179
4	295	1304	3368	5346	5634	4027	2037	778	238	65	13	3	23108
5	33	40	46	26	11	2	0	0	0	0	0	0	158
6	1	0	0	0	0	0	0	0	0	0	0	0	1
Total	354	1394	3475	5402	5656	4031	2037	778	238	65	13	3	23446

Table 2: Reduced graphs with no isolated vertex, $r = 4,5,6,7$

The list of adjacency matrices of the reduced graphs of rank up to 7 with no isolated vertices is available from the second author on request.

Using this data we can write down the formulae for the number $L(n,k)$ of labelled graphs of rank k on n vertices. As explained earlier, we need only consider graphs with no isolated vertex. We have

$$\begin{aligned} L(n,2) &= S(n+1,3), \\ L(n,3) &= S(n+1,4), \\ L(n,4) &= 28S(n+1,5)+180S(n+1,6)+420S(n+1,7), \\ L(n,5) &= 268S(n+1,6)+2520S(n+1,7)+9660S(n+1,8)+7560S(n+1,9). \end{aligned}$$

The expressions for $L(n,6)$ and $L(n,7)$ are implicit in our data but are cumbersome to write down. Table 3 gives some values.

We see, for example, that $L(n,3) > L(n,2)$ for all $n \geq 6$. The table suggests that perhaps $L(n,r)$ is an increasing function of r for $1 \leq r \leq n$ if $n \geq 8$.

$r \backslash n$	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0	0
2		1	6	25	90	301	966	3025
3			1	10	65	350	1701	7770
4				28	600	8120	89040	864948
5					268	8148	151508	2228688
6						15848	972944	36324456
7							880992	70339752
8								158666816
Total	1	2	8	64	1024	32768	2097152	268435456

Table 3: Labelled graphs of given rank

8 Some graph constructions

In this section we say a few words about line graphs, cartesian products, and categorical products.

8.1 Line graphs

Proposition 13 *If G is a connected graph with $|V(G)| \neq 4$, then $L(G)$ is reduced. Hence there are only finitely many line graphs with given rank r.*

Proof Suppose that two edges e_1 and e_2 of G have the same neighbours in $L(G)$. Then e_1 and e_2 are disjoint but every edge that meets e_1 also meets e_2 and *vice versa*. So $e_1 \cup e_2$ is a union of connected components.

In fact, we can obtain a much better bound for the order of a line graph of rank r than the exponential bound in the general case. Suppose that the adjacency matrix A of $L(G)$ has non-zero eigenvalues $\lambda_1, \ldots, \lambda_r$ (in decreasing order). We know that $\lambda_2, \ldots, \lambda_r \geq -2$; since the trace of A is zero, we have $\lambda_1 \leq 2(r-1)$. Now the number of edges of $L(G)$ is half the trace of A^2, which is not more than $2r(r-2)$. So the number of vertices of a line graph is bounded by a quadratic function of its rank.

In addition, we have:

Proposition 14 *If G is a connected graph with n vertices, then $L(G)$ has rank at least $n-2$.*

Proof G contains a spanning tree T, and $L(G)$ contains $L(T)$ as an induced subgraph, so that the rank of $L(G)$ is at least as great as that of $L(T)$; and the nullity of $L(T)$ is at most one [8].

This is best possible: if n is divisible by 4, then $L(C_n) = C_n$ has rank $n-2$.

We note as a curiosity that all the reduced graphs of rank at most 5 are line graphs.

8.2 Cartesian product

The vertex set of the Cartesian product $G_1 \square G_2$ is $V(G_1) \times V(G_2)$; there is an edge from (v_1, v_2) to (w_1, w_2) if and only if either $v_1 = w_1$ and v_2 is joined to w_2, or $v_2 = w_2$ and v_1 is joined to w_1.

Proposition 15 *If G_2 has no vertices of degree 0 or 1, and G_1 has no isolated vertices, then $G_1 \square G_2$ is reduced.*

Proof The vertex (v_1, v_2) has at least two neighbours (x_1, x_2) with $x_1 = v_1$, whereas the vertex (w_1, w_2) for $w_1 \neq v_1$ has at most one such neighbour. So equivalent vertices have the same first coordinate. But (v_1, v_2) has a neighbour (x_1, x_2) with $x_2 = v_2$, while (v_1, w_2) has no such neighbour if $w_2 \neq v_2$.

Now we have

$$A(G_1 \square G_2) = (A(G_1) \otimes I) + (I \otimes A(G_2)),$$

so the eigenvalues of $G_1 \square G_2$ are all sums of an eigenvalue of G_1 and an eigenvalue of G_2 (with appropriate multiplicities).

8.3 Categorical product

The vertex set of the categorical product $G_1 \times G_2$ is $V(G_1) \times V(G_2)$; there is an edge from (v_1, v_2) to (w_1, w_2) if and only if v_1 is joined to w_1 and v_2 is joined to w_2.

Proposition 16 *If G_1 and G_2 are reduced and have no isolated vertices then $G_1 \times G_2$ is reduced.*

Proof The neighbour set of (v_1, v_2) is the Cartesian product of the neighbour sets of v_1 and v_2, and so determines these two sets as long as both are non-empty.

9 Number of edges

The *Turán graph* $T_{n,r}$ is the complete r-partite graph on n vertices with parts as nearly equal as possible, that is, all parts of size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. *Turán's Theorem* asserts that it has the largest number of edges of any graph on n vertices containing no $(r+1)$-clique (see West [17], p. 208).

Now ${}^{r}T_{n,r} = K_r$, so $T_{n,r}$ has rank r; moreover, a graph of rank r cannot contain an $(r+1)$-clique. So we have:

Proposition 17 *The largest number of edges in a graph on n vertices of rank r is realised by the Turán graph.*

For the minimum we have the following:

Proposition 18 *(a) The smallest number of edges of a graph of rank $r > 1$ on n vertices (with $n \geq r$) is*

$$\begin{cases} r/2 & \text{if } r \text{ is even,} \\ (r+3)/2 & \text{if } r \text{ is odd.} \end{cases}$$

(b) The smallest number of edges of a connected graph of rank $r > 1$ on n vertices (with $n \geq r$) is

$$\begin{cases} n-1 & \text{if } r \text{ is even} \\ n & \text{if } r \text{ is odd.} \end{cases}$$

Proof (a) The extremal graph consists of $r/2$ disjoint edges if r is even, or $(r-3)/2$ disjoint edges and a triangle if r is odd. We cannot do better since there must be at least r non-isolated vertices; and if we have only $(r+1)/2$ edges in the case when r is odd, then some component would be a path of length 2, but this does not have full rank.

(b) If r is even, a path of length $r-1$ has rank r. (The rank of a tree is twice the size of the largest set of pairwise disjoint edges, see Theorem 8.1 on p. 233 of [5].) Now blow up an end vertex into a set of $n-r+1$ vertices. If r is odd, then no tree has rank r (since trees are bipartite and have even rank). But if we take a triangle with a "tail" of length $r-3$, we obtain a graph of rank r on n vertices (this follows easily from Problem 2b on p. 11 of Biggs [4]), and again we can blow up the end vertex of this graph to obtain a graph with n vertices.

10 Open problems

- Is it true that $m(r)$ is equal to the function given earlier, and that the only reduced graphs of rank r on $m(r)$ vertices are those produced by Constructions A and B?
- What is the largest order of a connected line graph of rank r? Earlier, we gave a quadratic upper bound for this. A linear lower bound $2(r-1)$ comes from the graph $K_2 \square K_{r-1} = L(K_{2,r-1})$.
- Is it true that the number $L(n,r)$ of labelled graphs on n vertices with rank r is an increasing function of r for $1 \le r \le n$, for all $n \ge 8$?
- We conjecture that $m(r)$ is strictly monotonic. We know that $m(r+2) \ge 2m(r)+1$, and obviously this conjecture would follow from our main conjecture about the value of $m(r)$. However, a direct proof presents some difficulties. It would suffice to show that, if G is a reduced graph on $m(r)$ vertices with rank r, then it is possible to add one vertex to G joined to a suitable set of existing vertices so that the rank increases by only one.

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