Problems on Discrete Metric Spaces

Edited by Peter J. Cameron
School of Mathematical Sciences
Queen Mary and Westfield College
London E1 4NS
U.K.

Abstract

These problems were presented at the Third International Conference on Discrete Metric Spaces, held at CIRM, Luminy, France, 15–18 September 1998. The names of the originators of a problem are given where these are known and different from the presenter of the problem at the conference.

Terminology: In a metric space, the point y is *between* x and z if $d(x, y) + d(y, z) = d(x, z)$. A *d-segment* is a set of the form $\langle x, z\rangle$ of points between x and z. A set S is *d-convex* if, whenever $x, z \in S$, then $\langle x, z\rangle \subseteq S$.

The *path metric* in a connected graph G is defined by the rule that $d(x, y)$ is the length of the shortest path from x to y.

1. Transformations which preserve d-convexity. Proposed by Petru Soltan.

Address: Usm, Faculty of Mathematics, A. Mateevich Str. 60, Kishinev MD-2009, Moldova
Email: `soltan@usm.md`

Let $\mathbb{R}^n$ be a real normed space of dimension n, endowed with the metric $d(x_1, x_2) = ||x_1 - x_2||$.

Problem: Find all affine transformations of $\mathbb{R}^n$ which preserve d-convexity.

Editor's Note: One could also ask: For which normed spaces is it true that all transformations which preserve d-convexity are affine? If every point on the boundary of the unit ball is an extreme point, then all such transformations are affine; and d-convexity coincides with convexity in the usual sense, so every affine transformation preserved d-convexity. In general, however, the 'd-convexity group' neither contains nor is contained in the affine group. For example, in the case of the l^1 norm on $\mathbb{R}^n$, this group is the *wreath product* of the homeomorphism group of $\mathbb{R}$ (the group of order-preserving and order-reversing permutations) with the symmetric group S_n.

A related question: For which normed spaces is it true that, for any points x and y, the set of points between x and y is d-convex? This is true in many spaces, including all those of dimension 2, but is false for the l^∞ norm on $\mathbb{R}^3$.

See the book by Soltan [17] for further details.

2. Avoiding facets of a zonotope. Proposed by Petru Soltan.
Address: Usm, Faculty of Mathematics, A. Mateevich Str. 60, Kishinev MD-2009, Moldova
Email: `soltan@usm.md`

Let Z^n be a *zonotope* in $\mathbb{R}^n$, that is,

$$Z^n = I_1 + I_2 + \cdots + I_k,$$

where $k \geq n$ and $I_1, I_2, \ldots, I_k$ are segments with a common endpoint. Note that Z^n and its facets are centrally symmetric.

Problem: Prove that there is a set of at most 2^{n-1} hyperplanes $\Gamma_1, \ldots, \Gamma_s$, each of which contains the central point of Z^n, such that for every $(n-1)$-dimensional facet F^{n-1} of Z^n, there exists $t \leq s$ such that $F^{n-1} \cap \Gamma_t = \emptyset$.

Proposer's note: The problem can be posed also for a *zonoid*, but the definition is very complicated, and it is necessary to introduce supplementary notions, including a special integral. See the book [2].

3. An approach to the partition problem. Proposed by Pierre Duchet.
Address: UPR 175 'Combinatoire' du CNRS, Univ. Paris 6, 4, Pl. Jussieu – Case 189, 75252 Paris Cedex 05, France
Email: `duchet@ecp6.jussieu.fr`

A set E, endowed with a closure mapping operating on E-subsets (simply called "sets" in the sequel) is called a *convexity space* if the closure of any set S, denoted by $\langle S\rangle$, is the set-union of the closures of finite S-subsets and if $\langle\emptyset\rangle=\emptyset$.

Define the *k-core* of a set S to be

$$\heartsuit_k(S) = \bigcap_{|A|=k} \langle S \setminus A \rangle.$$

Define the *k-crossing core* $K_k(S)$ to be the set of all *k-crossing points* of S, that is, points which lie in $\langle A_i \rangle$ for k pairwise disjoint S-subsets $A_1, \ldots, A_k$. The "partition conjecture" (see [8]) would be a consequence of the following conjectures.

Conjecture 1: Let k, r be integers not less than 2. If $K_2(R) \neq \emptyset$ for every r-element set R then $K_2(S) \cap \heartsuit_k(S) \neq \emptyset$ for every set S with $(k-1)(r-1)+1$ elements.

Conjecture 2: Under the same hypotheses, $K_2(S) \cap \heartsuit_k(S) \subseteq K_{k+1}(S)$.

4. Helly property for pairs. Proposed by Yuri Rabinovich.
Address: Dept. of Math. and CS, Ben-Gurion University of the Negev, Beer Sheva, Israel
Email: uri@cs.bgu.ac.il

Let P_1, P_2 be disjoint Euclidean planes. Consider the set of pairs $K = (K_1, K_2)$ with $K_i \subseteq P_i$ and K_i convex for $i = 1, 2$. A family $\mathcal{F}$ is *strongly intersecting* if there exists $p \in P_1 \cup P_2$ such that, for all $K = (K_1, K_2) \in \mathcal{F}$, either K_1 or K_2 contains p. (We will say that K *contains* p.) A family $\mathcal{F}$ is *weakly intersecting* if there exist $p_1 \in P_1$ and $p_2 \in P_2$ such that any $K \in \mathcal{F}$ contains either p_1 or p_2.

Problem: Assume that $\mathcal{F}$ has the property that any three of its elements are strongly intersecting. Is $\mathcal{F}$ weakly intersecting?

This is true with 4 replacing 3. The problem can be generalized by replacing two planes with any number of spaces of any dimension.

5. When is area a 2-metric? Proposed by Peter J. Cameron.
Address: School of Mathematical Sciences, Queen Mary and Westfield College, London E1 4N, U.K.
Email: p.j.cameron@qmw.ac.uk

Consider Heron's formula for the area of a Euclidean triangle:

$$\Delta(A, B, C) = \sqrt{s(s-a)(s-b)(s-c)},$$

where $d(B, C) = a$, $d(C, A) = b$, $d(A, B) = c$, and $s = (a + b + c)/2$. In a general metric space this is a totally symmetric function which vanishes if and only if one of A, B, C is between the others. When is it a 2-metric in the sense of Deza and Rosenberg [7]? That is, when does the *tetrahedron inequality*

$$\Delta(A, B, C) \leq \Delta(X, B, C) + \Delta(A, X, C) + \Delta(A, B, X)$$

hold? Perhaps the failure of this condition in some strong way implies that the space is hyperbolic in some sense. Similar questions can be asked for other 'area formulae', such as those in the elliptic or hyperbolic plane.

6. Uniqueness of the modular closure. Proposed by Alexander Karzanov.
Address: Institute for System Analysis, Russian Academy of Science, 9, Prospect 60 Let Oktyabrya, 117312 Moscow, Russia
Email: `sasha@cs.isa.ac.ru`

Let μ be a rational-valued metric on a finite set T. Is it true that:

(a) (T, μ) has precisely one modular closure (V, m), up to isomorphism;

(b) $|V|$ is bounded by a polynomial in $|T|$?

Here by a *modular closure* one means a metric space (V, m) which is an extension of (T, μ) (that is, $T \subseteq V$ and $m(x, y) = \mu(x, y)$ for all $x, y \in T$) such that (V, m) is modular (that is, each three elements of V have a median) and is obtained by the following process:

Initially set $V := T$ and $m := \mu$. Choose a medianless triple in V, $\{s_0, s_1, s_2\}$ say, and add a median v for it, by defining

$$m(s_i, v) = (m(s_i, s_j) + m(s_i, s_k) - m(s_j, s_k))/2,$$

where $\{i, j, k\} = \{0, 1, 2\}$. Then, considering an ordering $s_3, \ldots, s_n$ of the remaining elements of V, define

$$m(s_i, v) = \max\{m(s_i, s_j) - m(s_j, v) : j = 0, \ldots, i - 1\}$$

for each $i = 3, \ldots, n$. Update $V := V \cup \{v\}$ and iterate until each triple for the current (V, m) has a median.

A priori the modular closure may depend on the order in which the medianless triples are treated and the distances $m(s_i, v)$ are established. One easily shows that the process terminates in a finite number of iterations. It is known that (a) holds if the topological dimension of the *tight span* of μ is at most two: see Karzanov [11]. We conjecture that (b) holds for any finite rational metric μ, while (a) is false for some μ.

Additional question: Does the above process terminate in a finite number of iterations for any finite real-valued metric μ?

7. Representing all finite metrics. Proposed by Vincent Moulton.
Address: Physics and Math. Dept., Mid Sweden University, S 851-70 Sundsvall, Sweden
Email: `vincent.moulton@fmi.mh.se`

This problem is due to Jack Koolen and the proposer.

Let X be a finite set. Can we find a finite set P of pairwise linearly independent metrics on X, and a simplicial complex L on the vertex set P, such that

- every simplex of L is linearly independent, and
- for every metric d on X, there is a simplex D_d of L such that d is a linear combination with positive coefficients of the metrics in D_d?

Koolen and Moulton have given an affirmative answer for $|X| \leq 5$.

8. How many d-convex sets? Proposed by Victor Chepoi.
Address: LIM-CNRS, 163, Avenue de Luminy, 13288 Marseille Cedex 9, France
Email: `chepoi@lim.univ-mrs.fr`

This problem is due to V. Soltan [16].

Conjecture Every metric space (X, d) on n points possesses at least $3n-2$ d-convex sets.

A positive answer for path metrics of graphs was given by Soltan and Chepoi [18].

9. Compactness for bridged graphs? Proposed by Pierre Duchet.
Address: UPR 175 'Combinatoire' du CNRS, Univ. Paris 6, 4, Pl. Jussieu – Case 189, 75252 Paris Cedex 05, France
Email: `duchet@ecp6.jussieu.fr`

This problem is due to Hahn, Sands, Sauer and Woodrow [10].

A graph is *bridged* if it does not contain any isometric cycle of length at least 4. Prove that every cycle of a bridged graph G is contained in some finite induced bridged subgraph of G.

10. Basis graphs of matroids. Proposed by Victor Chepoi.
Address: LIM-CNRS, 163, Avenue de Luminy, 13288 Marseille Cedex 9, France
Email: `chepoi@lim.univ-mrs.fr`

This problem is due to S. Maurer [13].

Let $\mathcal{B}$ be the set of bases of a matroid $\mathcal{M}$. The *basis graph* of $\mathcal{M}$ has vertex set $\mathcal{B}$; vertices B_1 and B_2 are adjacent if and only if $|B_1 \triangle B_2| = 2$. Maurer proved that the graph G is the basis graph of a matroid if and only if it satisfies the conditions:

(a) If $d_G(u, v) = 2$, then

$$[u, v] = \{w : d_G(u, w) + d_G(w, v) = 2\}$$

induces either a square, a square pyramid, or an octahedron.

(b) If x, y, z, v induce a 4-cycle with $d_G(x, z) = d_G(y, v) = 2$, then for any vertex s, we have

$$d_G(s, x) + d_G(s, z) = d_G(s, y) + d_G(y, v).$$

(c) There is a vertex x_0 whose neighbourhood $N(x_0)$ is the line graph of a bipartite graph.

Conjecture: Condition (c) can be dropped.

11. Fully gated graphs. Proposed by Jean-Paul Doignon.
Address: ULB, Bd. du Triomphe, C.P. 216, B-1050 Bruxelles, Belgium
Email: `doignon@ulb.ac.be`

A graph $G = (V, E)$ is *fully gated* if every non-empty d-convex subset of V is gated. Here, a subset F of V is *gated* if

$$(\forall p \in V)(\exists g \in F)(\forall x \in F)\ d(p, x) = d(p, g) + d(g, x).$$

Problem: Determine the fully gated graphs.

Since (a) the cartesian product of graphs is fully gated if and only if the factors are fully gated, and (b) a graph is fully gated if and only if its blocks are fully gated, it suffices to determine those fully gated graphs which are 2-connected and indecomposable with respect to cartesian product. Such graphs include median graphs (for example, complete bipartite graphs), and dual polar spaces of type $D_n(q)$ (see [4]).

Another question is the complexity of the recognition problem for fully gated graphs.

12. Vertex-transitive graphs and Cayley graphs. Proposed by Wilfried Imrich.
Address: Angewandte Math., Montanuniv. Leoben, A-8700 Leoben, Austria
Email: `imrich@unileoben.ac.at`

This problem is due to W. Woess [20]

Problem: Is every locally finite, vertex-transitive graph *quasi-isometric* (in the sense of Gromov [9]) to a Cayley graph?

Results of Trofimov [19] imply a positive solution for graphs of polynomial growth. Also, graphs with infinitely many ends, all of which are thin, seem to present no problem.

13. Which fullerenes are halfcube-embeddable? Proposed by Michel Deza.
Address: LIENS-DMI, ENS-Paris, 45, Rue d'Ulm, 75005 Paris, France
Email: `deza@dmi.ens.fr`

A *fullerene* is a 3-valent polyhedron with only 6-gonal and (twelve) 5-gonal faces. Only four fullerenes (with 20, 26, 44, 80 vertices) are known to have skeletons which are isometrically embeddable into halved n-cubes (with $n = 10, 12, 16, 22$ respectively). Also, only four fullerenes (with 20, 28, 36, 60 vertices) are known to have the property that the skeletons of their duals are isometrically embeddable into halved n-cubes (with $n = 6, 7, 8, 10$ respectively). *Conjecture*: There are no other fullerenes with either of these properties.

(The *half-cube* consists of all binary words of length n and even weight, two vertices adjacent if their Hamming distance is 2. See [6].)

For the next two problems, we need some definitions. An *association scheme* on a finite set X is a partition of the set of 2-element subsets of X into classes $C_1, \ldots, C_d$ such that, given $x, y \in X$, the number of points z with $\{x, z\} \in C_i$ and $\{y, z\} \in C_j$ depends only on the class C_k containing $\{x, y\}$: this number is denoted by p_{ij}^k. Let A_i denote the adjacency matrix of the graph with edge set C_i. If the graph C_1 is connected and C_i is the set of pairs at distance i in this graph, the scheme is called *metric*; in this case, the graph C_1 is called *distance-regular*. If there exist polynomials $f_1, \ldots, f_d$ with $\deg(f_i) = i$ such that $A_i = f_i(A_1)$, the scheme is called *P-polynomial*. Recall the theorem of Delsarte [5]: *An association scheme is metric if and only if it is P-polynomial.*

14. Polynomials of hypermetric schemes. Proposed by Michel Deza.
Address: LIENS-DMI, ENS-Paris, 45, Rue d'Ulm, 75005 Paris, France
Email: `deza@dmi.ens.fr`

A n-point metric space is *hypermetric* if

$$\sum_{1\leq i<j\leq n} b_i b_j d(x_i, x_j) \leq 0$$

for all choices of $b_1, \ldots, b_n \in \mathbb{Z}$ with $\sum_{1\leq i\leq n} b_i = 1$. (For example, taking $b_i = b_j = 1$, $b_k = -1$ and all others zero, we obtain the triangle inequality. Taking three of b_i to be one, two to be minus one and all others zero, we obtain the 5-*gonal* inequality.) Koolen and Shpectorov [12, 15] have classified all hypermetric association schemes. How is the hypermetric property (or only the 3- and 5-gonal properties) reflected in the polynomials f_i of the scheme (which is P-polynomial by Delsarte's theorem above)?

15. Intersections of balls. Proposed by Vladimir Levenshtein.
Address: Institute of Applied Mathematics, Russian Academy of Science, Miusskaya Sq. 4, 125047 Moscow, Russia
Email: `leven@spp.keldysh.ru`

In a metric association scheme, fix t with $1 \leq t \leq d$ and set

$$N_k(t) = \sum_{i=0}^{t} \sum_{j=0}^{t} p_{ij}^k.$$

(This is the cardinality of the intersection of two balls of radius t whose centres have distance k.) One might expect that

$$N_1(t) \geq \ldots \geq N_d(t).$$

But this fails in some schemes, even those arising from distance-transitive graphs. *Problem*: Formulate general sufficient conditions for this monotonicity property to hold. (It is true in the Hamming and Johnson schemes, for example.)

16. Spectra of distance matrices. Proposed by Tom Hayden.
Address: Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA

Email: hayden@ms.uky.edu

This problem was proposed by Jim Wells who was unable to attend the conference. A *distance matrix* $D = (d_{ij})$ is a matrix whose entries $d_{ij} = d(p_i, p_j)^2$ are the squares of the distances between n points $p_1, \ldots, p_n$ in Euclidean space. *Problem*: Characterise the spectra of distance matrices. (It is known that, if $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of such a matrix, then

$$\lambda_1 > 0, \quad \lambda_i \leq 0 \text{ for } i > 1, \quad \sum_{j=1}^{n} \lambda_j = 0.$$

It is conjectured that these conditions are sufficient.)

17. Congruence criteria for finite subsets of quaternionic projective space. Proposed by Boumediene Et-Taoui.
Address: Dépt. Math., Univ. Mulhouse, France
Email: b.ettaoui@univ-mulhouse.fr

Blumenthal [1] proved:

- Let M_1, M_2 be two non-degenerate m-tuples of points of real projective space $\mathbb{R}P^n$. Then M_1 and M_2 are congruent if and only if all the corresponding sub-triples are congruent.

Brehm and Et-Taoui [3] proved:

- Let M_1, M_2 be two non-degenerate m-tuples of points of complex projective space $\mathbb{C}P^n$. Then M_1 and M_2 are congruent if and only if all the corresponding sub-quadruples are congruent.

The same authors *conjectured*:

- Let M_1, M_2 be two non-degenerate m-tuples of points of quaternionic projective space $\mathbb{H}P^n$. Then M_1 and M_2 are congruent if and only if all the corresponding sub-sextuples are congruent.

18. Random cyclic metric space? Proposed by Peter J. Cameron.
Address: School of Mathematical Sciences, Queen Mary and Westfield College, London E1 4N, U.K.
Email: p.j.cameron@qmw.ac.uk

A countable metric space admitting a cyclic automorphism permuting the points in a single cycle is specified by a function $f : \mathbb{N} \to \mathbb{R}_+$ which satisfies

$$\max_{x<n}\{|f(x) - f(n-x)|\} \leq f(n) \leq \min_{y<n}\{f(y) + f(n-y)\}.$$

(Identify the vertex set with $\mathbb{Z}$, and set $d(x,y) = f(|y-x|)$.) Assume that the values of $d(x,y)$ for $x \neq y$ lie in the set $\{1,2,\ldots,d\}$. If we choose $f(n)$ for $n = 1,2,\ldots$, the displayed inequalities never conflict. Suppose that we choose $f(n)$ randomly from the allowed values. What can be said about the resulting 'random cyclic metric space'? (For $d = 2$, we obtain the path metric in the countable random graph, or Rado's graph, R: see [14].)

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