Topology in permutation groups

Peter J. Cameron
Queen Mary, University of London
London E1 4NS
U.K.
p.j.cameron@qmw.ac.uk

Abstract

This paper is a survey of some topological aspects of permutation groups, especially concerning the question: "What does it tell us about a permutation group if it is a group of homeomorphisms of a topology?" This topic is not unrelated to a natural topology on the symmetric group, that of pointwise convergence; this connection is also discussed. The main new result of the paper is an elementary proof of part of a theorem of Macpherson and Praeger, according to which a group which preserves no non-trivial topology is highly transitive.

1 Introduction

In a lecture at Oberwolfach in the 1980s with the title "Topology in permutation groups", Helmut Wielandt defended the proposition that topology in permutation groups is of no use, and came to the conclusion that (with the addition of the word "almost") this was indeed the case.

Wielandt made it clear that he was not speaking about the topology *of* permutation groups. (There is a natural topology on the symmetric group, namely the topology of pointwise convergence. In the case of countable degree, the open subgroups are those lying between the pointwise and setwise stabiliser of a finite set, and the closed subgroups are the automorphism groups of first-order structures.)

Nevertheless, many in the audience felt that Wielandt's conclusion was too pessimistic. Since then, further results have supported this view. The present paper is a survey of some of these results. I begin with an account of Wielandt's own work on the subject, the connection of non-Hausdorff topologies preserved by G and the notions of primitivity and strong primitivity. The next topic is a theorem of Macpherson and Praeger, according to which a primitive group which preserves no non-trivial topology is highly transitive. Their proof uses some deep results from model theory. I give an elementary argument to replace part of the proof.

Some topologies low in the separation hierarchy can be interpreted as relational structures (specifically, preorders), so that their homeomorphism groups are closed subgroups of the symmetric group. This observation is a bridge to the next topic, the topology of permutation groups. This topic is too large for a survey, but some of the most important results relating to the earlier material are discussed.

By convention, a structure on a set Ω is *trivial* if its automorphism group is the full symmetric group on Ω. Clearly, knowing only that a group preserves a trivial structure tells us nothing about the group.

2 Primitivity and strong primitivity

A transitive permutation group G on a set Ω is *primitive* if G preserves no equivalence relation on Ω apart from the trivial ones (equality and the relation Ω^2).

In his pioneering study of infinite permutation groups, Wielandt [20] observed that this condition is not strong enough to derive analogues of results on finite primitive groups. Specifically, let G is a primitive group on a finite set Ω, and let Δ be a non-empty proper subset of Ω. Then the following separation property holds: for any distinct $\alpha, \beta \in \Omega$, there exists $g \in G$ with $\alpha g \in \Delta$ and $\beta g \notin \Delta$. This fails for infinite primitive groups: if G is the group of all order-preserving permutations of the real numbers, Δ the set of positive real numbers, and $\alpha < \beta$, then no such g can exist. Wielandt proposed the property of *strong primitivity* to remedy this defect. What follows is a slightly more general approach.

A *preorder* on a set Ω is a reflexive and transitive relation on Ω. A transitive permutation group G on Ω is called *strong* if every G-invariant preorder is symmetric (and so is an equivalence relation); and G is *strongly primitive* if it is strong and primitive, that is, the only G-invariant preorders on Ω are the trivial equivalence relations.

For any transitive permutation group G, there is a natural bijection between G-congruences and subgroups of G containing the point stabiliser G_α. Hence G is primitive if and only if G_α is a maximal proper subgroup of G. The analogue for strong primitivity is as follows:

Proposition 2.1 *Let G be a transitive permutation group on Ω. There is a natural bijection between G-invariant preorders on Ω and submonoids of G containing G_α. Hence G is strong if and only if any submonoid of G properly containing the point stabiliser G_α is a subgroup of G; and G is strongly primitive if and only if G_α is a maximal proper submonoid of G.*

Proof Suppose that $M(\alpha)$ is a submonoid of G containing G_α. By conjugation, we can define M_β for all $\beta \in \Omega$. Now define $\alpha \to \beta$ if there is an element of $M(\alpha)$ mapping α to β (equivalently, every element mapping α to β lies in $M(\alpha)$. This relation is a preorder. For suppose that $\alpha \to \beta$ and $\beta \to \gamma$. Choose $h \in M(\beta)$ with $\beta h = \gamma$. Then $h = g^{-1}kg$ for some $g, k \in M(\alpha)$ (where $\alpha g = \beta$); then $\alpha gh = \gamma$ and $gh = kg \in M(\alpha)$, so $\alpha \to \gamma$.

Conversely, if $\to$ is a G-invariant preorder, and

$$M(\alpha) = \{g \in G : \alpha \to \alpha g\},$$

then $M(\alpha)$ is a submonoid of G containing G_α; and the two constructions are mutually inverse.

For example, if G is strong, then $N_G(G_\alpha)/G_\alpha$ is a torsion group. In particular, the regular representation of G is strong if and only if G is a torsion group.

It is now a simple exercise to show that, if G is strongly primitive, then the separation property mentioned above holds.

3 Separation, preorders, and primitivity

If $\to$ is a preorder, then the relation $\equiv$ defined by

$$x \equiv y \text{ if and only if } x \to y \text{ and } y \to x$$

is an equivalence relation; and $\to$ induces a (partial) order on the set of equivalence classes of $\equiv$.

Given a topology $\mathcal{T}$ on Ω, construct a preorder on Ω by the rule that $x \to y$ if and only if every open set containing x contains y. Conversely, given a preorder $\to$, take the neighbourhoods

$$U_x = \{y \in \Omega : x \to y\} \qquad \text{for } x \in \Omega$$

as a base for a topology on Ω.

It is easy to see that the map

$$\text{preorder} \to \text{topology} \to \text{preorder}$$

is always the identity; the map

$$\text{topology} \to \text{preorder} \to \text{topology}$$

gives a stronger topology which is in general strictly stronger, though if Ω is finite, this map is also the identity. Note that the latter map is idempotent. We call a topology *relational* if it is fixed by this map (that is, if it is derived from a preorder).

By definition, G is strongly primitive if and only if it preserves no non-trivial preorder.

Theorem 3.1 *Let G be a transitive permutation group on Ω.*

(a) G is primitive if and only if every non-trivial topology invariant under G is T0.

(b) G is strongly primitive if and only if every non-trivial topology invariant under G is T1.

Proof We observe that the topology derived from a preorder $\to$ satisfies T0 if and only if the equivalence relation $\equiv$ is equality; and it satisfies T1 if and only if $\to$ has no arcs except loops. Hence, if G preserves a non-T0 topology, then it preserves a non-trivial equivalence, and if G preserves a non-T1 topology, then it preserves a preorder which is not symmetric.

The converse result in (a) is clear (if G is imprimitive, take open sets to be all unions of blocks). For (b), if G is not strongly primitive, then by definition it preserves a non-trivial preorder.

Remarks 1. If G preserves a non-T0 (resp. non-T1) topology, then it preserves a relational topology with the same property.

2. There is a unique countable homogeneous preorder which is universal (in the sense that it contains all finite or countable preorders) and homogeneous (in the sense that any isomorphism between finite sub-preorders extends to an automorphism). The corresponding relational topology is non-T0 and admits a transitive group of homeomorphisms. Similarly, there is a unique countable homogeneous partial order containing all finite or countable orders. These follow immediately from Fraïssé's construction method [5] and the fact that the classes of finite preorders and finite partial orders both have the amalgamation property.

4 Topologies and filters

Macpherson and Praeger [8] proved that a primitive permutation group on a countable set Ω which is not highly transitive (that is, is not n-transitive for some n) is contained in a maximal subgroup of $\mathrm{Sym}(\Omega)$. The main part of their argument was to show that such a group preserves a non-trivial filter on Ω. (A *filter* is a non-empty family $\mathcal{F}$ of subsets of Ω which is closed under finite intersections and closed under taking supersets, but does not contain the empty set. As usual, it is non-trivial if it is not invariant under $\mathrm{Sym}(\Omega)$. If Ω is countable, the only trivial filters are the *indiscrete filter* $\{\Omega\}$ and the *cofinite filter* consisting of all cofinite subsets.)

Their proof comes in two parts: first, prove that such a group preserves a non-trivial topology; and then show that any primitive group preserving a non-trivial topology must preserve a non-trivial filter. Their argument requires substantial machinery from model theory (the theorems of Ehrenfeucht–Mostowski, Engeler–Ryll-Nardzewski–Svenonius, and Cherlin–Harrington–Lachlan), and the connection between the filter and the topology is not clear. By contrast, the proof of the first step given below is more elementary, and only two constructions of filters from topologies are used: sets containing a finite intersection of open dense sets; and complements of finite unions of discrete sets.

The condition that G preserves a filter is a "strong condition", see Dixon, Neumann and Thomas [3], Macpherson and Neumann [7]. On the other hand, McDermott [6] has proved strong restrictions on permutation groups which preserve no non-trivial topology.

Remarks 1. I make essential use of Neumann's Separation Lemma [10, Lemma 2.3]: if G is a permutation group on Ω with no finite orbits, and Γ and Δ are finite subsets of Ω, then some translate of Γ by G is disjoint from Δ.

2. Primitivity is necessary. The group of all permutations preserving a partition of Ω into two parts of the same cardinality preserves a non-trivial topology but no non-trivial filter. Indeed, any imprimitive permutation group preserves a non-trivial topology, namely, the set of all unions of blocks in a given system of imprimitivity.

3. In some cases, one can see directly that a permutation group preserves a non-trivial topology. For example, from the countable random graph (see [2]) we derive a topology by taking the closed vertex neighbourhoods (consisting of a vertex v and all its neighbours) as a basis for the closed sets. In other cases, the construction of a topology is much less obvious.

Before stating the theorem, I give a simple characterisation of primitive automorphism groups of non-trivial topologies or filters. A *moiety* of the countable set Ω is an infinite subset of Ω whose complement is also infinite.

Lemma 4.1 *Let G be primitive on the countable set Ω.*

(a) G preserves a non-trivial topology if and only if there exists a moiety Δ of Ω such that, for all $g_1, \ldots, g_n \in G$, the set $\Delta g_1 \cap \ldots \cap \Delta g_n$ is empty or infinite.

(b) G preserves a non-trivial filter if and only if there exists a moiety Δ of Ω such that, for all $g_1, \ldots, g_n \in G$, the set $\Delta g_1 \cap \ldots \cap \Delta g_n$ is infinite.

Proof (a) Let G be primitive. Assume that G preserves a non-trivial topology. If there is a non-empty finite open set, take a minimal such set U; then U is a block of imprimitivity, necessarily a singleton, and the topology is discrete. So take U to be a non-empty non-cofinite open set; by what has just been proved, it is a moiety, and the condition of the lemma obviously holds.

Conversely, let Δ be such a set. Then the class of all non-empty sets of the form $\Delta g_1 \cap \ldots \cap \Delta g_n$, for $g_1, \ldots, g_n \in G$, is a basis for a non-trivial G-invariant topology on Ω.

(b) Neumann's Lemma implies that no filter invariant under an infinite transitive group can contain a finite set. So if G preserves a non-trivial filter, take U to be any non-cofinite set in the filter.

Conversely, if Δ is a moiety with the property of the lemma, then the class of all sets containing $\Delta g_1 \cap \ldots \cap \Delta g_n$ for some $g_1, \ldots, g_n \in G$ is a non-trivial filter admitting G.

Theorem 4.2 *Let Ω be an infinite set, and G a primitive permutation group on Ω. Then G preserves a non-trivial topology on Ω if and only if G preserves a non-trivial filter on Ω.*

Proof I begin with some remarks on topologies admitting primitive groups.

If such a topology contains a non-empty finite open set, then it is discrete: for the minimal open sets form a system of imprimitivity, and so all singletons are open. Similarly, if the topology contains a cofinite open set, then every cofinite set is open (but we cannot conclude that the topology is trivial in this case).

A set is dense if it meets every non-empty open set. Any finite intersection of dense open sets is non-empty. (The proof is by induction on the number of sets. Given dense open sets $U_1, \ldots, U_n$, the set $U_1 \cap \ldots \cap U_{n-1}$ is non-empty and open, by the inductive hypothesis; so it meets the dense open set U_n.) So the family of sets containing finite intersections of dense open sets is a filter.

If a topology admits a primitive group, and is the union of finitely many discrete subsets, then it is itself discrete. (To show this, it suffices to find a finite open set. Let $\Omega = X_1 \cup \ldots \cup X_n$, where $X_1, \ldots, X_n$ are pairwise disjoint discrete sets. Choose $x_1 \in X_1$. There exists an open set U_1 with $U_1 \cap X_1 = \{x_1\}$. If $U_1 = \{x_1\}$, we are done; so suppose not. Then, without loss, $U_1 \cap X_2$ is non-empty; choose $x_2 \in U_1 \cap X_2$. Then there is an open set U_2 with $U_2 \cap X_2 = \{x_2\}$, so that

$$\{x_2\} \subseteq (U_1 \cap U_2) \cap (X_1 \cup X_2) \subseteq \{x_1, x_2\}.$$

Continuing in this way, we end up with a finite open set.)

It follows that, if the topology is not discrete, then the set of complements of finite unions of discrete sets is a filter.

Now we begin the proof of the theorem. One direction is elementary: if $\mathcal{F}$ is a filter, then $\{\emptyset\} \cup \mathcal{F}$ is a topology, which is non-trivial if $\mathcal{F}$ is. So we suppose that G is primitive and preserves a non-trivial topology $\mathcal{T}$, and (for a contradiction) that G preserves no non-trivial filter. By our two constructions above, it follows that every open dense set is cofinite, and every discrete set is finite.

Form a graph Γ on Ω by joining x and y whenever there exist disjoint open sets containing x and y.

Case 1 Γ contains no infinite clique.

In this case, every finite clique is contained in a finite maximal clique.

Let $\{x_1, \ldots, x_n\}$ be a maximal clique. We can find pairwise disjoint open sets $U_1, \ldots, U_n$ with $x_i \in U_i$ for $i = 1, \ldots, n$. (For let W_{ij} and W_{ji} be disjoint open sets containing x_i and x_j respectively, for $i \neq j$; then put $U_i = \bigcap_{j \neq i} W_{ij}$.) Now $U = U_1 \cup \ldots \cup U_n$ is open and dense (by the maximality of the clique $\{x_1, \ldots, x_n\}$); so $F = \Omega \setminus U$ is finite.

Let $\{x'_1, \ldots, x'_m\}$ be another maximal clique, and define $U'_1, \ldots, U'_m, F'$ analogously. By Neumann's Separation Lemma, we may assume that the finite sets $\{x_1, \ldots, x_n\} \cup F$ and $\{x'_1, \ldots, x'_m\} \cup F'$ are disjoint.

Suppose that $m > n$. Then $x_1, \ldots, x_n$ lie in the union of the pairwise disjoint sets $U'_1, \ldots, U'_m$; say $x_i \in U'_{k_i}$ for $i + 1, \ldots, n$. Then one of the sets, say U'_l, does not contain any of $x_1, \ldots, x_n$. But then $U_1 \cap U'_{k_1}$, $\ldots$, $U_n \cap U'_{k_n}$ are pairwise disjoint open sets containing $x_1, \ldots, x_n$, and U_l is disjoint from all of them, a contradiction. So $m > n$ is impossible, and dually $n > m$ is impossible. Thus, we have $m = n$. We conclude that all maximal cliques have the same size n.

Now the induced subgraph of Γ on $U_1 \cup \ldots \cup U_n$ is complete n-partite. (It clearly contains the complete multipartite graph; but any additional edge would produce a clique of size greater than n.)

We claim that Γ is complete n-partite. For suppose not. Since complete n-partite graphs are characterised by their finite subgraphs, there exists a finite subset Z of Ω on which the induced subgraph is not complete n-partite. By Neumann's Separation Lemma, we may assume that $Z \cap F = \emptyset$. Then $Z \subseteq U_1 \cup \ldots \cup U_n$, and we have a contradiction. So the claim is proved.

Now, if $n > 1$, we have a contradiction to the primitivity of G, since G preserves the complete n-partite graph Γ. On the other hand, if $n = 1$, then Γ is a null graph, and so any two non-empty open sets intersect. So the class of sets which contain a non-empty open set is a filter, and is non-trivial since the topology is.

Case 2 Γ contains an infinite clique Y. Then the induced topology on Y is Hausdorff, and so Y contains an infinite discrete subset (see below). So the filter of complements of discrete sets is non-trivial.

The fact that an infinite Hausdorff space contains an infinite discrete subset is an exercise in the book by Sierpiński [17]. Since this is not very helpful, I include the proof. Let X be an infinite Hausdorff space. Call a point $x \in X$ *bad* if every open set containing x is cofinite. Clearly there is at most one bad point. (Bad points can exist; for example, the added point in the one-point compactification of an infinite discrete space.) Now define a sequence $x_1, x_2, \ldots$ of points of X and a decreasing sequence $Y_0, Y_1, Y_2, \ldots$ of infinite sets as follows:

- $Y_0 = X$.
- Given Y_{n-1}, let x_n be any non-bad point of Y_{n-1}, U_n an open set containing X_n such that $Y_n = Y_{n-1} \setminus U_n$ is infinite.

Now we claim that $Z = \{x_1, x_2, \ldots\}$ is discrete. For any x_n, we have an open set U_n containing x_n and no x_m for $m > n$. Also, by the Hausdorff property, there is an open set V_n containing x_n and no x_m for $m < n$. Then $Z \cap (U_n \cap V_n) = \{x_n\}$, and so Z is discrete.

Problem Can this result be quantified? That is, if X is a sufficiently vast Hausdorff space, must it contain an uncountable discrete set? The space $\mathbb{R}$ shows that cardinality $2^{\aleph_0}$ is not enough.

The companion theorem of Macpherson and Praeger [8] is:

Theorem 4.3 *Let Ω be countable, and assume that the permutation group G on Ω preserves no non-trivial filter on Ω. Then G is highly transitive (that is, G is n-transitive for all natural numbers n).*

I have been unable to produce an elementary proof of this theorem. In a special case, this can be done. Let G be any permutation group on Ω. The *algebraic closure* of a finite subset X of Ω is the union of the finite orbits of the pointwise stabiliser of X in G. We say that *algebraic closure is trivial* if the algebraic closure of X is precisely X, for all finite sets X.

Proposition 4.4 *Let G be a permutation group on Ω, for which algebraic closure is trivial. Then either G preserves a non-trivial topology on Ω, or G is highly transitive.*

Proof Suppose that algebraic closure is trivial, and that G is not highly transitive. Then there exists a finite set X whose pointwise stabiliser has more than one orbit outside X, and all its orbits are infinite. Let Y be one of these orbits. Then, for any elements $g_1, \ldots, g_n$ of G, the set $Yg_1 \cap \ldots \cap Yg_n$ is fixed by the pointwise stabiliser of the finite set $F = Xg_1 \cup \ldots \cup Xg_n$, and contains no point of F. So $Yg_1 \cap \ldots \cap Yg_n$ is empty or infinite. Now the result follows from Lemma 4.1.

One would expect that, if algebraic closure were non-trivial, one could use it to construct a non-trivial topological closure operation. But I have not succeeded in doing this. In particular, defining a set to be closed if it contains the algebraic closure of each of its finite subsets does not work.

5 Automorphism groups of filters

In this section we assume that the set Ω is countable. We show that any filter has a large automorphism group, and in particular, the automorphism group of an ultrafilter is a maximal subgroup of $\mathrm{Sym}(\Omega)$. This section is based on the work of Neumann [12], see also [7].

A filter $\mathcal{F}$ on Ω is *principal* if there is a subset X of Ω (necessarily non-empty) such that $\mathcal{F}$ consists of all subsets of Ω containing X. An *ultrafilter* is a maximal filter. Thus, a principal ultrafilter consists of all sets which contain a specified point $x \in \Omega$.

Given a filter $\mathcal{F}$, we let $A(\mathcal{F})$ denote the automorphism group of $\mathcal{F}$, and $B(\mathcal{F})$ the set of all permutations $g \in \mathrm{Sym}(\Omega)$ with the property that $\mathrm{Fix}(g) \in \mathcal{F}$, where $\mathrm{Fix}(g)$ is the set of fixed points of g. If $\mathcal{F}$ is the principal filter defined by X, then $A(\mathcal{F})$ and $B(\mathcal{F})$ are the setwise and pointwise stabiliser of X respectively. We will see that they behave in a similar way in general.

Proposition 5.1 *(a) For any filter $\mathcal{F}$, $B(\mathcal{F})$ is a normal subgroup of $A(\mathcal{F})$.*

(b) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters with $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $B(\mathcal{F}_1) \le B(\mathcal{F}_2)$.

(c) *If $\mathcal{F}$ is a non-principal ultrafilter then $A(\mathcal{F}) = B(\mathcal{F})$ is a highly transitive maximal subgroup of* $\mathrm{Sym}(\Omega)$ *which acts transitively on the sets of $\mathcal{F}$ which are moieties (that is, are not cofinite).*

Proof All is straightforward. For (a), if $X \in \mathcal{F}$ and $g \in B(\mathcal{F})$, then $Xg \supseteq X \cap \mathrm{Fix}(g)$, so $Xg \in \mathcal{F}$. So $B(\mathcal{F}) \subseteq A(\mathcal{F})$. If $g, h \in B(\mathcal{F})$, then $\mathrm{Fix}(gh^{-1}) \supseteq \mathrm{Fix}(g) \cap \mathrm{Fix}(h)$, so $gh^{-1} \in B(\mathcal{F})$; so this set is a group. Normality is clear.

Part (b) is trivial. For (c), we show first that $A(\mathcal{F}) = B(\mathcal{F})$. So let g be any permutation in $A(\mathcal{F})$. Assume that $g \notin B(\mathcal{F})$, so that $\mathrm{Fix}(g) \notin \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, $\Omega \setminus \mathrm{Fix}(g) \in \mathcal{F}$, so there exists $h \in B(\mathcal{F})$ with $\mathrm{Fix}(h) = \Omega \setminus \mathrm{Fix}(g)$. Then $gh \in A(\mathcal{F})$ has no fixed points.

Now let X consist of alternate points from the cycles of gh (where, if a cycle has finite odd length, we leave one gap of length 3), $Y = Xgh$, and $Z = \Omega \setminus (X \cup Y)$. The three sets X, Y, Z comprise a partition of Ω, so one of them lies in the ultrafilter $\mathcal{F}$. But each is disjoint from its image under gh, so $\emptyset \in \mathcal{F}$, a contradiction.

To prove the high transitivity, we use the fact that a non-principal ultrafilter $\mathcal{F}$ contains the filter $\mathcal{F}_c$ of cofinite sets; and $B(\mathcal{F}_c)$ is the finitary symmetric group. (Indeed, if $\mathcal{F}$ is any filter containing all the cofinite sets, then $B(\mathcal{F})$ is highly transitive.)

Let X and Y be moieties belonging to $\mathcal{F}$. Then $X \cap Y$ is an infinite set in $\mathcal{F}$. Partition this set into infinite subsets A and B. Since $\mathcal{F}$ is an ultrafilter, one of A and B is in $\mathcal{F}$, say A; then the pointwise stabiliser of A contains a permutation mapping X to Y.

Finally, let G be a subgroup of $\mathrm{Sym}(\Omega)$ which properly contains $A(\mathcal{F})$. Then G must move some moiety in $\mathcal{F}$ to one not in $\mathcal{F}$; hence G acts transitively on moieties. Since the pointwise stabiliser of any moiety in G is the symmetric group on its complement, it follows easily that $G = \mathrm{Sym}(\Omega)$.

6 Maximal subgroups

If M is a maximal proper subgroup of $\mathrm{Sym}(\Omega)$, where Ω is countable, then clearly either M contains the finitary symmetric group F, or $MF = \mathrm{Sym}(\Omega)$. In the latter case, M has countable index in $\mathrm{Sym}(\Omega)$ (since F is countable), and so by Theorem 8.1 below, M is the stabiliser of a finite set. Conversely, the stabiliser of a finite set is a maximal subgroup. In the former case, M is highly transitive. More generally, Macpherson and Neumann [7] proved that any supplement for F in $\mathrm{Sym}(\Omega)$ lies between the pointwise and setwise stabiliser of a finite set.

As noted earlier, Macpherson and Praeger [8] proved:

Theorem 6.1 *A primitive subgroup of* $\mathrm{Sym}(\Omega)$ *which is not highly transitive is contained in a maximal subgroup of* $\mathrm{Sym}(\Omega)$.

Not every proper subgroup of $\mathrm{Sym}(\Omega)$ is contained in a maximal subgroup. Like every uncountable group, $\mathrm{Sym}(\Omega)$ is the union of a chain of subgroups, none of which are maximal. Macpherson and Neumann proved that any chain of subgroups with union $\mathrm{Sym}(\Omega)$ has length strictly greater than $|\Omega|$. It follows that some subgroup in the chain necessarily contains the finitary symmetric group.

7 The rational world

Among countable topologies, there is one which is particularly interesting: the set $\mathbb{Q}$ of rational numbers, with the usual topology. Unlike the case of $\mathbb{R}$, the topology of $\mathbb{Q}$ does not determine the order up to reversal; indeed, $\mathbb{Q}$ is homeomorphic to $\mathbb{Q}^n$ for any positive integer n, and its homeomorphism group is highly transitive. More generally, Sierpiński [16] showed that a countable topological space is homeomorphic to $\mathbb{Q}$ if and only if it is totally disconnected and $T1$ and without isolated points. Neumann [11] calls such a space a *rational world*. He has investigated the rational world and its homeomorphism group, and among other things has determined all the possible cycle structures of homeomorphisms. Strikingly, there are $2^{\aleph_0}$ non-conjugate homeomorphisms which permute all the points in a single cycle.

The most dramatic theorem on the rational world is due to Mekler [9], with an alternative proof by Truss [19]. It states that, for countable groups, our necessary and sufficient conditions of Lemma 4.1 for a group to act on a topology are actually necessary and sufficient for it to act on the rational world:

Theorem 7.1 *Let G be a countable permutation group on a countable set Ω. Then the following are equivalent:*

(a) G is permutation-isomorphic to a subgroup of the group of homeomorphisms of $\mathbb{Q}$;

(b) There is a subset Δ of Ω such that, for any elements $g_1, \ldots, g_n \in G$, the set $\Delta g_1 \cap \cdots \cap \Delta g_n$ is empty or infinite.

8 Topology of permutation groups

The symmetric group itself carries a natural topology, the topology of pointwise convergence. A basis of neighbourhoods of the identity is formed by the pointwise stabilisers of finite sets of points.

The basic facts about this topology are well known; this section contains only a few brief remarks. For the remainder of this section, we assume that Ω is countable. In this case, $\mathrm{Sym}(\Omega)$ is metrisable: taking $\Omega = \mathbb{N}$, we may set $d(g,h) = 1/n$ if $ig = ih$ and $ig^{-1} = ih^{-1}$ for all $i < n$ but one of these equations fails for $i = n$, where $g, h \in \mathrm{Sym}(\Omega)$, $g \neq h$.

The topology of $\mathrm{Sym}(\Omega)$ can be recovered from the group structure. This is a consequence of the following theorem due to Semmes [13] and Dixon, Neumann and Thomas [3]:

Theorem 8.1 *Let Ω be countable. Then a subgroup of index smaller than $2^{\aleph_0}$ in $\mathrm{Sym}(\Omega)$ contains the pointwise stabiliser of a finite subset X of Ω, and is contained in the setwise stabiliser of X.*

The first assertion means that the open subgroups of $\mathrm{Sym}(\Omega)$ are precisely those of countable index. We say that $\mathrm{Sym}(\Omega)$ has the *small index property.* The two assertions together are called the *strong small index property.*

Closed subgroups of $\mathrm{Sym}(\Omega)$ are characterised by the next result. Recall that a structure is *homogeneous* if every isomorphism between finite substructures can be extended to an automorphism.

Proposition 8.2 *The following conditions are equivalent for a subgroup G of $\mathrm{Sym}(\Omega)$, where Ω is countable:*

(a) G is closed in $\mathrm{Sym}(\Omega)$;

(b) G is the automorphism group of a first-order structure on Ω;

(c) G is the automorphism group of a homogeneous relational structure on Ω.

Now we can ask whether the topology of a closed subgroup of $\mathrm{Sym}(\Omega)$ can be recovered from its group structure, that is, whether the group has the small index property. This has been shown in many cases (for example, projective and affine spaces over finite fields, the countable random graph, the countable atomless Boolean algebra), but is not true for all closed subgroups (see Evans and Hewitt [4] for a counterexample).

9 Higher cardinalities

The questions considered here can of course be generalised to larger infinite sets. Typically, positive theorems tend to take on the more uncertain status of conjectures or independence results. I refer interested readers to the papers [1, 14, 15, 18].

References

[1] J. E. Baumgartner, S. Shelah, and S. Thomas, Maximal subgroups of infinite symmetric groups, *Notre Dame J. Formal Logic* **34** (1993), 1–11.

[2] P. J. Cameron, The random graph, pp. 331–351 in *The Mathematics of Paul Erdős*, (ed. J. Nešetřil and R. L. Graham), Springer, Berlin, 1996.

[3] J. D. Dixon, P. M. Neumann and S. Thomas, Subgroups of small index in infinite symmetric groups, *Bull. London Math. Soc.* **18** (1986), 580–586.

[4] D. M. Evans and P. R. Hewitt, Counterexamples to a conjecture on relative categoricity, *Annals of Pure and Applied Logic* **46** (1990), 201–209.

[5] R. Fraïssé, Sur certains relations qui généralisent l'ordre des nombres rationnels, *C. R. Acad. Sci. Paris* **237** (1953), 540–542.

[6] J. P. J. McDermott, personal communication.

[7] H. D. Macpherson and P. M. Neumann, Subgroups of infinite symmetric groups, *J. London Math. Soc.* (2) **42** (1990), 64–84.

[8] H. D. Macpherson and C. E. Praeger, Maximal subgroups of infinite symmetric groups, *J. London Math. Soc.* (2) **42** (1990), 85–92.

[9] A. H. Mekler, Groups embeddable in the autohomeomorphisms of $\mathbb{Q}$, *J. London Math. Soc.* (2) **33** (1986), 49–58.

[10] P. M. Neumann, The lawlessness of finitary permutation groups, *Arch. Math.* **26** (1975), 561–566.

[11] P. M. Neumann, Automorphisms of the rational world, *J. London Math. Soc.* (2) **32** (1985), 439–448.

[12] P. M. Neumann, personal communication.

[13] S. Semmes, Endomorphisms of infinite symmetric groups, *Abstracts Amer. Math. Soc.* **2** (1981), 426.

[14] S. Shelah and S. Thomas, Implausible subgroups of infinite symmetric groups, *Bull. London Math. Soc.* **20** (1988), 313-318.

[15] S. Shelah and S. Thomas, Subgroups of small index in infinite symmetric groups, II, *J. Symbolic Logic* **54** (1989), 95-99.

[16] W. Sierpiński, Une propriété topologique des ensembles dénombrables denses en soi, *Fund. Math.* **1** (1920), 11–16.

[17] W. Sierpiński, *General Topology* (translated by C. C. Krieger), University of Toronto Press, Toronto, 1956.

[18] S. Thomas, Aspects of infinite symmetric groups, in *Infinite groups and group rings* (ed. J. M. Corson textitet al.), World Scientific, Singapore, 1993, pp. 139–145.

[19] J. K. Truss, Embeddings of infinite permutation groups, *Proceedings of Groups – St. Andrews 1985* (ed. E. F. Robertson and C. M. Campbell), London Math. Soc. Lecture Notes **121**, Cambridge University Press, Cambridge, 1986, 335–351.

[20] H. Wielandt, *Unendliche Permutationsgruppen*, (notes prepared by A. Mader), Math. Inst. Univ. Tübingen 1960; English translation *Infinite Permutation Groups* by P. V. Bruyns, pp. 199–235 in H. Wielandt, *Mathematische Werke*, Volume 1: Group Theory (ed. B. Huppert and H. Schneider), de Gruyter, Berlin, 1994.