Partitions and permutations

Peter J. Cameron

School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK

Abstract

With any permutation g of a set Ω is associated a partition of Ω into the cycles of g. What information do we get about a group G of permutations if we know either the set or the multiset of partitions of Ω, or of partitions of $n = |\Omega|$, which arise as the cycle partitions of its elements? Some partial answers to these questions are given.

Key words: permutation group, cycle, partition, cycle index, Parker vector

1 Introduction

Let Ω be a finite set of cardinality n. To any permutation g of Ω, we associate the partition $\mathrm{CP}(g)$ of Ω into cycles of g, and the partition $\mathrm{cp}(g)$ of the integer n given by the cycle lengths.

Now let G be a group of permutations of Ω (a subgroup of the symmetric group on Ω). We are interested in the question: What information about G is contained in the set or multiset of cycle partitions corresponding to its elements? We can distinguish eight questions:

(a) We can ask for a characterisation of the sets or multisets of partitions arising from groups, or we can ask to what extent knowledge of the set or multiset determines the group.
(b) We can consider set partitions $\mathrm{CP}(G) = \{\mathrm{CP}(g) : g \in G\}$, or number partitions $\mathrm{cp}(G) = \{\mathrm{cp}(g) : g \in G\}$.
(c) We can consider $\mathrm{CP}(G)$ or $\mathrm{cp}(G)$ as sets, or as multisets. (In the latter case, for each partition π of Ω or n, we know the number of permutations $g \in G$ for which $\mathrm{CP}(g)$ or $\mathrm{cp}(g)$ is equal to π.)

The recognition problem (the first question in (a)) is complicated by the fact that, while permutation groups can be represented efficiently (a subgroup of

S_n can be generated by $O(n)$ elements, and hence is specified by $O(n^2 \log n)$ bits), no such short representation of sets of partitions is available. P. Kolaitis [13] has suggested the *polynomial delay* model as a way round this problem:

Problem 1 *(a) Is there an algorithm which takes as input a subset X of P_n, performs a polynomial-time computation after reading each partition in X, and decides whether $X = \mathrm{CP}(G)$ for some subgroup G of S_n?*

(b) Is there a polynomial-length 'certificate' S for X (for example, a generating set for G) with the property that, given X, on performing a polynomial-time computation after reading each partition in X, we can confirm that $X = \mathrm{CP}(G)$?

Certainly an affirmative solution to (a) would settle (b) also; these questions are analogous to the definitions of the classes P and NP respectively. Note that a small generating set for the group generated by an arbitrary set of permutations can be computed with polynomial delay: see Jerrum [12].

For the remainder of the paper, I will concentrate on the question of the extent to which cycle partitions determine the group. These will be considered for set partitions and for number partitions in the next two sections. Then there is a brief survey of the Parker vector of a permutation group, whose kth component is the number of orbits of the group on the set of its k-cycles. The last two sections discuss infinite versions of the results and some possible further directions.

2 Set partitions

In this section, we consider the question: What information about G do we obtain from $\mathrm{CP}(G)$? In particular, if $\mathrm{CP}(G_1) = \mathrm{CP}(G_2)$, must G_1 and G_2 be isomorphic, or even equal? The answers to the strongest conjectures are negative, as the following examples show. The first example is due to Alberto Leporati [14], who suggested this question to me; the third is due to Eamonn O'Brien [15].

Example 2 *Let $n = 5$, and let G_1 be the cyclic group of order 5, acting regularly. Then $\mathrm{CP}(G_1)$ consists of the trivial partitions of Ω (into a single part, and into parts of size 1). If $\mathrm{CP}(G_1) = \mathrm{CP}(G_2)$, then clearly G_2 is also a cyclic group of order 5, but it could be any of the six such subgroups of S_5. Note, however, that these subgroups are all conjugate in S_5, so G_1 and G_2 are isomorphic as permutation groups.*

Example 3 *Let G be the group $C_7 \times C_7$ of order 49. We embed G in the symmetric group of degree 28 by choosing four subgroups of order 7, say*

$P_1, \dots, P_4$, and taking the union of the coset spaces G/P_i (with regular action) for $i = 1, \dots, 4$. It is easy to see that the automorphism group $\mathrm{GL}(2,7)$ *of G does not act transitively on the collection of 4-sets of proper subgroups of G; choosing the set of four subgroups from different orbits gives rise to permutation groups which are not isomorphic. However,* $\mathrm{CP}(G)$ *consists of the partition with 28 parts of size 1, the four partitions with seven parts of size 1 (in one orbit) and three of size 7 (each with multiplicity 6), and the partition into the four orbits (with multiplicity 24), in either case. So* $\mathrm{CP}(G)$ *(even as multiset) does not determine G up to permutation isomorphism.*

Example 4 *Eamonn O'Brien [15] found two pairs of examples of groups G_1, G_2 with* $\mathrm{CP}(G_1) = \mathrm{CP}(G_2)$ *for which G_1 and G_2 are not isomorphic as abstract groups. The pairs are numbers 19 and 111, and numbers 94 and 249, in the lists of groups of order 64 contained in the computer systems GAP and MAGMA (see [6] and [2]), acting in faithful permutation representations of least possible degree 16.*

For example, the first two groups are generated by the permutations

$$
\begin{aligned}
&(1,11,3,10,2,12,4,9)(5,16,8,14,6,15,7,13),\\
&(1,8,4,6,2,7,3,5)(9,13,11,15,10,14,12,16),\\
&(1,3,2,4)(5,7,6,8)(9,11,10,12)(13,15,14,16),\\
&(1,3,2,4)(5,8,6,7)(9,11,10,12)(13,16,14,15),\\
&(1,4,2,3)(5,8,6,7)(9,11,10,12)(13,15,14,16),\\
&(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)
\end{aligned}
$$

and

$$
\begin{aligned}
&(1,12,4,10,2,11,3,9)(5,16,8,14,6,15,7,13),\\
&(1,7,3,6,2,8,4,5)(9,14,12,15,10,13,11,16),\\
&(5,6)(7,8)(13,14)(15,16),\\
&(1,3,2,4)(5,7,6,8)(9,12,10,11)(13,16,14,15),\\
&(1,4,2,3)(5,8,6,7)(9,12,10,11)(13,16,14,15),\\
&(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)
\end{aligned}
$$

respectively; and $\mathrm{CP}(G_1)$ *and* $\mathrm{CP}(G_2)$ *consist of the same set of twelve partitions of* $\{1, \dots, 16\}$ *(with the same multiplicities).*

So $\mathrm{CP}(G)$ *does not determine G up to isomorphism, even if G is transitive.*

However, the set $\mathrm{CP}(G)$ does determine a lot of information about G, as the next results show. We define $\mathrm{OP}(G)$ to be the set of all partitions of Ω which are orbit partitions corresponding to subgroups of G. Thus, $\mathrm{CP}(G)$ is a subset of $\mathrm{OP}(G)$.

Theorem 5 *Let G_1 and G_2 be permutation groups on Ω satisfying $\mathrm{CP}(G_1) = \mathrm{CP}(G_2)$. Then*

(a) $|G_1| = |G_2|$;
(b) $\mathrm{OP}(G_1) = \mathrm{OP}(G_2)$;
(c) $\mathrm{CP}(G_1)$ *and* $\mathrm{CP}(G_2)$ *are identical as multisets; that is,*

$$|\{g \in G_1 : \mathrm{CP}(g) = \pi\}| = |\{g \in G_2 : \mathrm{CP}(g) = \pi\}|$$

for all partitions π of Ω.

PROOF. (a) The proof is by induction on the degree $n = |\Omega|$. Note that $\mathrm{CP}(G)$ determines the orbit αG for $\alpha \in \Omega$, since

$$\alpha G = \{\beta \in \Omega : (\exists \pi \in \mathrm{CP}(G))(\alpha \text{ and } \beta \text{ are in the same part of } \pi)\}.$$

Also, if G_α is the stabiliser of α in G, then $\mathrm{CP}(G)$ determines

$$\mathrm{CP}(G_\alpha) = \{\pi \in \mathrm{CP}(G) : \{\alpha\} \text{ is a part of } \pi\}.$$

By induction, $\mathrm{CP}(G_\alpha)$ determines $|G_\alpha|$. Since

$$|\alpha G| \cdot |G_\alpha| = |G|$$

by Lagrange's Theorem, the result is proved.

We have incidentally shown that, if $\mathrm{CP}(G_1) = \mathrm{CP}(G_2)$, then G_1 and G_2 have the same orbits on Ω.

(b) It is clear that the orbit partition of a subgroup H is the supremum (in the lattice of partitions of Ω) of the set of cycle partitions of the elements of H (or, indeed, of a generating set for H).

(c) Let π be any partition of Ω, and let $G_1(\pi)$ be the subgroup of G_1 consisting of all permutations fixing all the parts of π (the intersection of G with the corresponding *Young subgroup* of the symmetric group: see Fulton [5]). Now $\mathrm{CP}(G_1(\pi))$ consists of those partitions in $\mathrm{CP}(G_1)$ lying below π in the partition lattice. Hence, if $G_2(\pi)$ is analogously defined in G_2, we have $\mathrm{CP}(G_1(\pi)) = \mathrm{CP}(G_2(\pi))$, and hence $|G_1(\pi)| = |G_2(\pi)|$, by part (a) of the theorem.

Now the conclusion follows by Möbius inversion, since an element $g \in G_1$ satisfies $\mathrm{CP}(g) = \pi$ if and only if $g \in G_1(\pi)$ but $g \notin G_1(\sigma)$ for any σ strictly below π in the partition lattice. $\square$

The last part of the theorem shows that, in this case, the set and the multiset of cycle partitions carry exactly the same information. From the first part, we deduce:

Corollary 6 *If H is a subgroup of G satisfying either* $\mathrm{CP}(G) = \mathrm{CP}(H)$ *or* $\mathrm{OP}(H) = \mathrm{OP}(G)$*, then $G = H$.*

PROOF. The conclusion is immediate from the theorem if $\mathrm{CP}(G) = \mathrm{CP}(H)$. A very similar argument shows that, if $\mathrm{OP}(G_1) = \mathrm{OP}(G_2)$, then $|G_1| = |G_2|$, and again it follows that if $H \leq G$ satisfies $\mathrm{OP}(H) = \mathrm{OP}(G)$, then $H = G$. $\square$

See [1] for an application of these results to the power of database query languages.

Two problems which are left open between the conclusions of the theorem and the preceding examples are the following:

Problem 7 *Let G_1 be a permutation group which is either (a) primitive, or (b) regular. Let G_2 be another permutation group on the same set, satisfying* $\mathrm{CP}(G_1) = \mathrm{CP}(G_2)$*. Are G_1 and G_2 isomorphic as permutation groups?*

Note that, if G_1 is regular and $\mathrm{CP}(G_1) = \mathrm{CP}(G_2)$, then G_2 is also regular, and the subgroup lattices of G_1 and G_2 are isomorphic.

In some special cases, more can be said. Let us say that a group G_1 is *cycle-determined* if $\mathrm{CP}(G_1) = \mathrm{CP}(G_2)$ implies $G_1 = G_2$.

Proposition 8 *(a) A group of order 2 is cycle-determined.*
(b) A regular non-cyclic elementary abelian group is cycle-determined.

PROOF. (a) An involution is determined by its cycle partition.

(b) The parts of the partitions in $\mathrm{CP}(G)$, for such a group G, are the lines of an affine space over $\mathrm{GF}(p)$, and G is the translation group of this space. $\square$

In fact, if $\mathrm{CP}(G_1) = \mathrm{CP}(G_2)$, then every involution in G_1 is also in G_2, and *vice versa*; in particular, *a group generated by involutions is cycle-determined.*

More generally, we say that H is 1-*closed* in G if any permutation in G preserving the H-orbits lies in H. Now, if $H \leq G_1$ and H is 1-closed in G_1, then the number of permutations preserving the orbit partition of H is the same in G_2 as in G_1, and so $H \leq G_2$. Hence we have:

Proposition 9 *If G is generated by subgroups which are cycle-determined and 1-closed in G, then G is cycle-determined.* $\square$

3 Number partitions; cycle index

The information contained in the multiset $\mathrm{cp}(G)$ is precisely the (Pólya) *cycle index* of G, the polynomial in indeterminates $s_1, \ldots, s_n$ given by

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{n} s_i^{c_i(g)},$$

where $c_i(g)$ is the number of i-cycles of g. So we immediately deduce from Theorem 5(c):

Theorem 10 *If $\mathrm{CP}(G_1) = \mathrm{CP}(G_2)$, then $Z(G_1) = Z(G_2)$.* $\square$

The role of the cycle index in combinatorial enumeration is well-known (see for example, [11]). Note in particular:

Proposition 11 *$Z(G)$ determines the number of orbits of G on Ω^k and on the set of k-element subsets of Ω. In particular, it determines $|G|$.* $\square$

The last statement follows from the first, but is most easily seen by substituting $s_1 = 1$, $s_i = 0$ for $i > 1$, in $Z(G)$.

The *permutation character* of G is the function $\pi : G \to \{0, 1, \ldots, n\}$ given by

$$\pi(g) = \text{number of fixed points of } g$$

for $g \in G$. A well-known Möbius inversion shows that the permutation character of G determines $Z(G)$: for

$$\pi(g^k) = \sum_{i|k} i c_i(g),$$

and so

$$c_k(g) = \frac{1}{k} \sum_{i|k} \mu(k/i) \pi(g^i).$$

Hence different actions of the same group which have the same permutation character will have the same cycle index. The simplest example is given by the two actions of the Klein group V_4 on six points:

$$
\begin{aligned}
G_1 &= \{1, (1,2)(3,4), (1,2)(5,6), (3,4)(5,6)\},\\
G_2 &= \{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}.
\end{aligned}
$$

A more elaborate example, given by Guralnick and Saxl [9], has G_1 primitive and G_2 imprimitive.

Groups with the same cycle index which are not even be abstractly isomorphic are not difficult to find. For example, if two groups G_1 and G_2 have the same numbers of elements of each order then their regular representations have the same cycle index. Examples include

$$
\begin{array}{lll}
Q_8 \times C_2 & \text{and } C_4 \times C_4 & (n = 16),\\
C_3 \times C_3 \times C_3 & \text{and } P_{27} & (n = 27),
\end{array}
$$

where P_{27} is the nonabelian group of order 27 and exponent 3. Of course, the groups of Example 4 also have the same cycle index.

Turning to the set $\mathrm{cp}(G)$ without multiplicities, clearly little can be said. This information determines the orders of elements of G (and hence the exponent of G) but not $|G|$ or the number of orbits of G.

4 The Parker vector

In the preceding sections, we treat the cycles of a permutation as the parts of a partition, ignoring the cyclic structure on each. A complementary approach is to consider the cycles in isolation. This approach was taken by Parker [16].

Let $\Gamma_i(G)$ be the set of all i-cycles which appear in the cycle decomposition of some element of G, and let $\Gamma(G) = \bigcup_{i=1}^{n} \Gamma_i(G)$. Now G act on $\Gamma_i(G)$ by conjugation: let $p_i(G)$ be the number of orbits in this action. Note that $\Gamma_1(G)$ is isomorphic to Ω as G-set, since each point of Ω occurs as a singleton cycle in the identity. Parker gave the formula

$$p_i(G) = \frac{1}{|G|} \sum_{g \in G} i\gamma_i(G),$$

and deduced that

$$\sum_{i=1}^{n} p_i(G) = n$$

for any permutation group G of degree n.

For a summary of the information about G which is contained in its *Parker vector*

$$p(G) = (p_1(G), \ldots, p_n(G))$$

see Gewurz [7]. Note that $p(G)$ does not determine $|G|$; for example, we have $p(G_1 \wr G_2) = p(G_2 \wr G_1)$ for any two permutation groups G_1, G_2, where $\wr$ denotes the wreath product. On the other hand, Gewurz showed that some groups are determined by their Parker vectors, for example, the symmetric group S_n for $n \neq 6$. (More precisely, if G has degree n and Parker vector $(1, 1, \ldots, 1)$, then either $G = S_n$, or $n = 6$ and $G = \mathrm{PGL}(2, 5)$.)

Parker showed that, if G is the Galois group of a given polynomial of degree n over the rational numbers, then there is a randomised algorithm which computes efficiently the Parker vector of G.

Gewurz [8] pointed out that the Parker vector is determined by the cycle index (and so *a fortiori* by $\mathrm{CP}(G)$):

Proposition 12 $p_i(G) = i \dfrac{\partial}{\partial s_i} Z(G)|_{s_j=1}$. $\Box$

We can regard $\Gamma(G)$ as a set of permutations in the symmetric group S_n, where each point outside the cycle is regarded as being fixed. In general, of course, $\Gamma(G)$ is not a subset of G. In fact, the following holds (see [3]):

Proposition 13 *We have $\Gamma(G) \subseteq G$ if and only if G is a direct product of symmetric groups and cyclic groups of prime order.* $\Box$

In fact, if we set $C_0(G) = G$ and $C_{n+1}(G) = \langle \Gamma(C_n(G) \rangle$ for all $n \geq 0$, then $C_3(G) = C_4(G)$ for every finite permutation group G. There exist groups with $C_2(G) \neq C_3(G)$; such groups are p-groups for some odd prime p, but the problem of determining them is unsolved.

Problem 14 *Determine the finite permutation groups G with $C_2(G) \neq C_3(G)$.*

5 The infinite

Some of the above discussion extends to infinite groups. For the most part, counting results fail; some other results remain true, though more ingenuity is required for the proof.

Recall that a permutation group G on an infinite set Ω is

- *highly transitive* if it is n-transitive for all natural numbers n;
- *oligomorphic* if it has only finitely many orbits on Ω^n for all natural numbers n;
- *finitary* if all its elements move only finitely many points;
- *cofinitary* if all its non-identity elements fix only finitely many points.

(For further details, see [4].)

Theorem 15 *If* $\mathrm{CP}(G_1) = \mathrm{CP}(G_2)$ *and* G_1 *is* k*-transitive (resp. highly transitive, oligomorphic, finitary, cofinitary), then so is* G_2*.*

PROOF. For 'finitary' and 'cofinitary', this is obvious.

As in the proof of Theorem 5, $\mathrm{CP}(G)$ determines the orbits of G and also determines $\mathrm{CP}(G_\alpha)$ for all $\alpha \in \Omega$. By induction it determines the orbits of all n-point stabilisers. Now G is oligomorphic (resp. highly transitive) if the stabiliser of any n-tuple has only finitely many (resp. just one) orbit on the remaining points. The argument for k-transitivity is similar. $\square$

If G is oligomorphic, then a *modified cycle index* $\tilde{Z}(G)$ can be defined for G as follows: take the sum of the cycle indices of the groups G^Δ_Δ (the permutation group induced on Δ by its setwise stabiliser), where Δ runs over a set of representatives of the orbits of G on finite sets. This plays a similar role in enumeration to the cycle index of a finite permutation group. Indeed, oligomorphic permutation groups are precisely those for which such an enumeration theory can be developed. To make a connection with the work of Roland Fraïssé [10], note that if the oligomorphic group G is the automorphism group of a homogeneous structure M, then its modified cycle index is obtained by summing the cycle indices of the (unlabelled) structures in the age of M.

Does $\mathrm{CP}(G)$ determine the modified cycle index of the oligomorphic group G? For each finite set Δ, it determines $\mathrm{CP}(G^\Delta_\Delta)$, and hence $Z(G^\Delta_\Delta)$, by Theorem 10. However, $\mathrm{CP}(G)$ does not determine the orbits of G on Ω^n, or on the set of finite subsets of Ω, as Example 2 shows. Nevertheless, we have:

Theorem 16 *If* $\mathrm{CP}(G_1) = \mathrm{CP}(G_2)$ *and* G_1 *is oligomorphic, then* $\tilde{Z}(G_1) = \tilde{Z}(G_2)$*.*

PROOF. Although we cannot determine the orbits, we can determine a set of orbit representatives for G on Ω^n, as follows: first determine orbit representatives for G on Ω; then, for each such α_1, determine orbit representatives for G_{α_1} on the points different from α_1; then for each such pair (α_1, α_2), determine orbit representatives for $G_{\alpha_1\alpha_2}$ on the points different from α_1 and α_2; and so on.

Now let $(\alpha_1, \dots, \alpha_n)$ be an orbit representative, and let $\Delta = \{\alpha_1, \dots, \alpha_n\}$. We can determine $Z(G_\Delta^\Delta)$. However, the orderings of a given set Δ lie in $n!/|G_\Delta^\Delta|$ different orbits on n-tuples. So we multiply $Z(G_\Delta^\Delta)$ by this factor, and we sum these terms over all orbit representatives on n-tuples. Finally, we sum the result for all n. $\square$

The original question of Leporati which motivated this study is open in the infinite case:

Problem 17 *If $H \leq G$ and $\mathrm{CP}(H) = \mathrm{CP}(G)$, does it follow that $H = G$?*

Recall that a *base* for a permutation group G on Ω is a sequence B of elements of Ω whose pointwise stabiliser is the identity. The answer to Problem 17 is affirmative if G has a finite base. This follows easily by induction from the fact that, if $H \leq G$ and, for some point $\alpha \in \Omega$ we have $\alpha^G = \alpha^H$ and $G_\alpha = H_\alpha$, then $H = G$. For the statement that $\alpha^G = \alpha^H$ means that H contains a set of coset representatives for G_α in G.

More generally, if $H \leq G$ and $\mathrm{CP}(G) = \mathrm{CP}(H)$, then the *closures* of G and H in the symmetric group (in the topology of pointwise convergence) are equal. For the closure of G consists of all permutations which preserve every G-orbit on n-tuples for all n. By hypothesis, a set of H-orbit representatives on n-tuples is also a set of G-orbit representatives; so G and H have the same orbits on n-tuples, and hence the same closure. (This is a generalisation of the preceding remark, since it is known that a group having a finite base is closed.)

Since every finite permutation group is closed (and has a finite base), this produces an alternative proof of Corollary 6.

6 Remarks and problems

I have said nothing about the problem of deciding whether a given set S of partitions is equal to $\mathrm{CP}(G)$ for some group G. Clearly this could be done by computing (as in Theorem 5(c)) the number of permutations $g \in G$ with $\mathrm{CP}(g) = \pi$ for each partition $\pi \in S$, choosing in all ways the correct number of permutations, and checking whether we have a group. This algorithm will be very inefficient!

A subset of $\mathrm{CP}(G)$ may suffice to specify G. The *stabiliser chain* of G relative to a base B of size k is the sequence $(G_0 = G, G_1, \ldots, G_{k-1}, G_k = 1)$ of subgroups, where G_i is the stabiliser of the first i points of B. A set $S \subseteq G$ is a *strong generating set* (relative to B) if it contains a subset S_i which generates G_i for $i = 0, 1, \ldots, k$. I owe the following result to Leonard Soicher [17].

Proposition 18 *If S is a strong generating set for G, then $\mathrm{CP}(S)$ determines $|G|$.*

PROOF. Suppose that the subset $X = \mathrm{CP}(S)$ of P_n is given. If we know B, we argue by induction exactly as in the proof of Theorem 5(a). We have to modify the condition for two points α and β to be in the same orbit: this holds if and only if there is a sequence

$$\alpha = \omega_0, \omega_1, \ldots, \omega_d = \beta$$

such that, for $i = 1, \ldots, d$, there is a partition in X for which ω_{i-1} and ω_i lie in the same part.

If we do not know B, we test all tuples $B = (\alpha_1, \ldots, \alpha_k)$ for which the partition of B into singletons is not contained in any non-trivial partition of X, applying the preceding algorithm to B. If B is a base such that X comes from a strong generating set relative to B, we obtain $|G|$. If not, then we obtain a smaller number. So the largest number obtained is $|G|$. It suffices to test all tuples which are minimal with respect to the specified property. $\square$

I do not know how to recognise the set $\mathrm{CP}(S)$ arising from a strong generating set S for a permutation group.

We could play the same game with subsets instead of partitions. That is, we could assume that we know the cycles, or the supports of the cycles, or just the cycle lengths, either with or without multiplicities, and ask what information we can deduce about the group. Note that, with or without multiplicities, the following implications hold between what we can learn about a group from this information:

$$\begin{array}{ccccc} & & \text{set partitions} & \rightarrow & \text{number partitions} \\ & & \downarrow & & \downarrow \\ \text{cycles} & \rightarrow & \text{cycle supports} & \rightarrow & \text{cycle lengths} \end{array}$$

I leave to the diligent reader the task of carrying out such an investigation, starting with the following facts:

(a) the multiset of cycle lengths determines $|G|$ and the Parker vector of G;
(b) the set of cycles determines the groups $C_n(G)$ for $n \geq 1$ (see Section 4);
(c) the set of cycle supports determines the orbits of G.

References

[1] P. Bonizzoni, G. Della Vedova, A. Leporati, D. Lucarella and G. Mauri, A unifying framework to characterize the power of query languages, preprint.

[2] W. Bosma and J. Cannon, *The MAGMA Handbook*, available from the Computational Algebra Group, University of Sydney, 1999.

[3] P. J. Cameron, Cycle-closed permutation groups, *J. Algebraic Combinatorics* **5** (1996), 315–322.

[4] P. J. Cameron, *Permutation Groups*, Cambridge University Press, Cambridge, 1999.

[5] W. Fulton, *Young Tableaux*, London Math. Soc. Student Texts **35**, Cambridge University Press, Cambridge, 1997.

[6] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.2; Aachen, St Andrews, 1999.
(http://www-gap.dcs.st-and.ac.uk/~gap)

[7] D. A. Gewurz, Reconstruction of permutation groups from their Parker vectors, *J. Group Theory*, to appear.

[8] D. A. Gewurz, Parker vectors and cycle indices of permutation groups, in preparation.

[9] R. M. Guralnick and J. Saxl, Primitive permutation characters, pp. 364–367 in *Groups, Combinatorics and Geometry* (ed. M. W. Liebeck and J. Saxl), London Math. Soc. Lecture Notes **165**, Cambridge University Press, Cambridge, 1992.

[10] R. Fraïssé *Theory of Relations*, North-Holland, Amsterdam, 1986.

[11] F. Harary and E. M. Palmer, *Graphical Enumeration*, Academic Press, New York, 1973.

[12] M. R. Jerrum, A compact representation for permutation groups, *J. Algorithms* **7** (1986), 60–78.

[13] P. Kolaitis, personal communication.

[14] A. Leporati, personal communication.

[15] E. O'Brien, personal communication.

[16] R. A. Parker, personal communication.

[17] L. H. Soicher, personal communication.