Orbital chromatic and flow roots

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Abstract

The chromatic polynomial $P_\Gamma(x)$ of a graph Γ is a polynomial whose value at the positive integer k is the number of proper k-colourings of Γ. If G is a group of automorphisms of Γ, then there is a polynomial $OP_{\Gamma,G}(x)$, whose value at the positive integer k is the number of orbits of G on proper k-colourings of Γ.

It is known there are no real negative chromatic roots, but they are dense in $[\frac{32}{27}, \infty)$. Here we discuss the location of real orbital chromatic roots. We show, for example, that they are dense in $\mathbb{R}$, but under certain hypotheses, there are zero-free regions. Our hypotheses include parity conditions on the elements of G and also some special types of graphs and groups.

We also look at orbital flow roots. Here things are more complicated because the orbit count is given by a multivariate polynomial; but it has a natural univariate specialisation, and we show that the roots of these polynomials are dense in the negative real axis.

1 Introduction

The chromatic polynomial $P_\Gamma(x)$ of a graph Γ is the polynomial whose value at the positive integer k is the number of proper k-colourings of Γ. (By convention, if Γ contains a loop, then $P_\Gamma(x) = 0$.)

Let G be a group of automorphisms of Γ. For each element $g \in G$, the number of proper k-colourings of Γ fixed by g is equal to $P_{\Gamma/g}(k)$, where Γ/g denotes the graph obtained by shrinking each vertex cycle of the permutation g to a single vertex, putting a loop at that vertex if the cycle contains an edge, and putting an edge between two vertices if there is an edge in Γ between the corresponding cycles of g. (For a colouring is fixed by g if and only if it is constant on the vertex cycles of g, and any such colouring induces a colouring of Γ/g; conversely, any proper colouring of Γ/g lifts to a colouring of Γ fixed by g. If a cycle of g contains an edge, then no proper colouring is fixed by g.)

By the Orbit-Counting Lemma, the number of G-orbits on proper k-colourings is

$$\frac{1}{|G|}\sum_{g\in G} P_{\Gamma/g}(k).$$

So, if we define the *orbital chromatic polynomial* of Γ and G to be

$$OP_{\Gamma,G}(x) = \frac{1}{|G|}\sum_{g\in G} P_{\Gamma/g}(x),$$

then substituting k into this polynomial gives the number of G-orbits on proper k-colourings of G.

In this paper, we will investigate whether and to what extent the known results on real zeros of the chromatic polynomial extend to the orbital chromatic polynomial. We begin with two simple observations:

Proposition 1.1 *(a) A positive integer is a root of $OP_{\Gamma,G}(x) = 0$ if and only if it is a root of $P_\Gamma(x) = 0$.*

(b) If G is the trivial group, then $OP_{\Gamma,G} = P_\Gamma$; so every chromatic root is an orbital chromatic root.

Proof (a) A graph has no orbits on proper k-colourings if and only if it has no proper k-colourings!

(b) Clear.

It is known that there are intervals of $\mathbb{R}$ containing no chromatic roots. We show first that no such result holds for orbital chromatic roots in general, and then that zero-free intervals do exist provided that certain parity conditions on the elements of G are satisfied, or for some special graphs and groups.

In the final section of the paper, we look at orbital flow roots, and prove some similar results.

2 Orbital chromatic roots

2.1 Orbital chromatic roots are dense in $\mathbb{R}$

For information about chromatic roots, we refer to the survey paper by Jackson [3].

The chromatic polynomial of a loopless graph has no negative real roots, and no roots in the interval $(0, 1)$. Jackson [2] showed that there are no roots in the interval $(1, \frac{32}{27}]$, and Thomassen [5] showed that this is best possible:

roots of chromatic polynomials are dense in $[\frac{32}{27}, \infty)$. Moreover, Sokal [4] showed that complex chromatic roots are dense in the whole complex plane.

Our first result asserts that there are no zero-free intervals for orbital chromatic roots. We begin by observing that such roots can be negative. Let Γ be the null graph on n vertices, and G the symmetric group S_n. Then orbits of G on k-colourings of Γ correspond to choices of n colours from the set of k, where repetitions are allowed and the order of the choices is not significant. From elementary combinatorics, this number is $\binom{k+n-1}{n}$. So we have

$$OP_{\Gamma,G}(x) = \frac{1}{n!}x(x+1)\cdots(x+n-1),$$

with roots $0, -1, \ldots, -(n-1)$.

Theorem 2.1 *Orbital chromatic roots are dense in* $\mathbb{R}$.

Proof First we consider the graph Γ consisting of m disjoint triangles, where the ith triangle has vertex set x_i, y_i, z_i. The group G will be the symmetric group S_m, permuting the m triangles (and preserving the three sets $\{x_1, \ldots, x_m\}$, $\{y_1, \ldots, y_m\}$, and $\{z_1, \ldots, z_m\}$).

The number of colourings of a triangle with k colours is $k(k-1)(k-2) = p(k)$, say. Now a proper colouring of Γ consists simply of a selection of m colourings, one for each triangle (with repetitions allowed); and counting orbits of G means that the order of the selection is not significant. By the elementary combinatorial result cited earlier, the number of colourings is

$$\frac{1}{m!}p(k)(p(k)+1)\cdots(p(k)+m-1) = \binom{p(k)+m-1}{m}.$$

That is, the orbital chromatic polynomial is

$$OP_{\Gamma,G}(x) = \binom{p(x)+m-1}{m}, \text{ where } p(x) = x(x-1)(x-2).$$

This shows that the roots of $f_r(x) = x(x-1)(x-2) + r = 0$ are orbital chromatic roots, for any non-negative integer r. Clearly $f_r(x)$ has a unique negative root α_r for $r > 0$, and $\alpha_r - \alpha_{r+1} \to 0$ as $r \to \infty$. (For $\alpha_r = -m$ when $r = m(m+1)(m+2)$, and so the number of roots between consecutive negative integers $-m$ and $-m-1$ increases with m. Moreover, the second derivative of $f_r(x)$ is negative for negative x, so the gaps between successive α_r decrease as r increases.)

Finally, choose a positive integer s, and let Γ_s be the sum of Γ and a complete graph on s vertices (that is, we take the disjoint union of these graphs, with all possible edges between them). It is clear that in any colouring of Γ_s, we must use s distinct colours for K_s, leaving $k-s$ colours for the remainder. So

$$OP_{\Gamma_s,G}(x) = x(x-1)\cdots(x-s+1)OP_{\Gamma,G}(x-s).$$

So every number of the form $\alpha_r + s$ is an orbital chromatic root. It is clear that these numbers are dense in $\mathbb{R}$.

2.2 Parity and zero-free intervals

However, under parity conditions, we do get zero-free intervals for orbital chromatic roots. To prove these, we need a slight refinement of the result about zero-free intervals: see [3, Theorem 3].

Proposition 2.2 *Let the graph Γ have v vertices, c connected components and b blocks. Then, for real x, the sign of $P_\Gamma(x)$ is $(-1)^v$ for $x < 0$; $(-1)^{v-c}$ for $0 < x < 1$, and $(-1)^{v-c-b}$ for $1 < x \leq \frac{32}{27}$.*

Now any automorphism induces permutations not only on the set of vertices of Γ, but also on the set of connected components.

Theorem 2.3 *Let G be a group of automorphisms of the loopless graph Γ. Let V and C be the sets of vertices and connected components of Γ.*

(a) If all elements of G are even permutations of V, then $OP_{\Gamma,G}(x)$ has no negative real roots.

(b) If all elements of G are even permutations of $V \cup C$, then $OP_{\Gamma,G}(x)$ has no roots in $(0,1)$.

Proof (a) The parity of a permutation g on n points is that of $n-k$, where k is the number of cycles of g. So, if every element of G is an even permutation on vertices, then the number of vertices of Γ/g is congruent mod 2 to the number v of vertices of Γ. So the sign of $P_{\Gamma/g}(x)$ is $(-1)^v$ for $x < 0$ (unless Γ/g has a loop, in which case it is zero). To find $OP_{\Gamma,G}(x)$ for negative x, we have to average $P_{\Gamma/g}(x)$ over the group; all these terms have the same sign or are zero, and not all are zero. So the average cannot be zero.

(b) is proved in the same way, using the parity on $V \cup C$ instead of that on V.

Continuing the example before Theorem 2.1, let Γ be the null graph on n vertices and G the alternating group A_n. Then

$$OP_{\Gamma,G}(x) = \frac{1}{n!}\left(x(x+1)\cdots(x+n-1) + x(x-1)\cdots(x-n+1)\right).$$

If n is even, this polynomial has only even powers of x and has positive coefficients, so it has no real roots at all. If n is odd, it has a single real root with multiplicity 1 at the origin.

It might be thought that, if all elements of G are even permutations of $V \cup C \cup B$, where B is the set of blocks, then $OP_{\Gamma,G}(x)$ would have no roots in $(1, \frac{32}{27}]$. But the argument fails, since the number of blocks of Γ/g is not always equal to the number of cycles of g on blocks of Γ. For example, let Γ be the complete bipartite graph $K_{2,p}$ and G the cyclic group C_p, where p is an odd prime. For any non-identity element $g \in G$, Γ/g is a path of length 2. Indeed, $OP_{\Gamma,G}(x) = x(x-1)f_p(x)/p$, where

$$f_p(x) = (x-2)^p + (x-1)^{p-1} + (p-1)(x-1).$$

For $p \geq 5$, we have $f_p(1) < 0$ and $f_p(\frac{32}{27}) > 0$, so there is an orbital chromatic root in $(1, \frac{32}{27})$. Indeed, f_5 has a root at approximately 1.12683.

2.3 Some special graphs

For the purposes of this section, we say that a graph Γ is a *special* (n,m) *graph* if there are two sets A and B of vertices with the properties that

- A is an independent set of size n;
- B is a clique of size m;
- for every vertex $v \in A$, the set of neighbours of v is B.

Such a graph has a group of automorphisms isomorphic to the symmetric group S_n permuting the vertices in A and fixing all other vertices.

Now we prove the following theorem:

Theorem 2.4 *Let Γ be a special (n,m) graph and $G = S_n$. Assume that $n - m \leq 1$. Then there is a graph Γ_{eq} with the property that*

$$OP_{\Gamma,G}(x) = \frac{1}{|G|} P_{\Gamma_{eq}}(x).$$

Hence $OP_{\Gamma,G}(x)$ has no negative real roots (and none in the intervals $(0,1)$ or $(1, \frac{32}{27}]$).

Proof We outline two proofs of this theorem, in the hope that both techniques will be useful in further work on this topic.

First proof: Let $\Gamma_0 = \Gamma \setminus A$. Choose any proper colouring of Γ_0 with k colours. Then m colours are required for the vertices in B. So we can extend this colouring to a colouring of A in $\binom{k-m+n-1}{n}$ ways, since the colours of vertices of A are unrestricted and (because of the group) order is not significant. So we have

$$OP_{\Gamma,G}(x) = \frac{1}{n!} P_\Gamma(x)(x-m)\cdots(x-m+n-1).$$

(This formula is correct for any values of n and m.)

Now let Γ_{eq} be obtained from Γ by adding an independent set $B^* = \{b_0, \ldots, b_{n-1}\}$, where b_i is joined to $m-i$ vertices of A. (This is possible since $m-n+1 \geq 0$.) Clearly

$$P_{\Gamma_{eq}}(x) = P_\Gamma(x)(x-n)\cdots(x-n+m-1),$$

so that Γ_{eq} is the required graph.

Second proof: We have included this proof because it shows that, at least sometimes, deletion-contraction can be used for orbital polynomials. The proof is by induction on n. For $n = 1$, the group is trivial, and the theorem holds for $\Gamma_{eq} = \Gamma$. So assume the result for special $(n-1, m-1)$ graphs, and let $\Gamma = \Gamma(n,m)$ be a special (n,m) graph.

Recall the *Stirling numbers of the first kind*: $(-1)^{n-k}s(n,k)$ is the number of permutations of $\{1, \ldots, n\}$ with k cycles. Thus we see that there are $|s(n,k)|$ permutations $g \in S_n$ for which $\Gamma(n,m)/g \cong \Gamma(k,m)$. Thus

$$OP_{\Gamma(n,m),G}(x) = \frac{1}{n!} \sum_{k=1}^{n} |s(n,k)| P_{\Gamma(k,m)}(x).$$

Let $\Gamma(n,m,r)$ denote the graph obtained from $\Gamma(n,m)$ by deleting one edge incident with each of r vertices in the coclique B, and set $[n,r] = P_{\Gamma(n,m,r)}(x)$, for $0 \leq r \leq n$.

We use the deletion-contraction identity

$$P_\Gamma(x) = P_{\Gamma\setminus e}(x) - P_{\Gamma/e}(x).$$

If e is an edge incident with a vertex of B of degree m in $\Gamma = \Gamma(n,m,r)$, then $\Gamma\setminus e \cong \Gamma(n,m,r+1)$ and $\Gamma/e \cong \Gamma(n-1,m,r)$. So we have

$$[n,r] = [n,r+1] - [n-1,r].$$

for $r < n$. By induction, we find that

$$[n,0] = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} [n-i, n-i-1].$$

Hence the polynomial we require is

$$\begin{aligned} S &= \sum_{k=0}^{n-1} (-1)^k s(n, n-k)[n-k, 0] \\ &= \sum_{k=0}^{n-1} \sum_{i=0}^{n-k-1} (-1)^{k+i} s(n, n-k) \binom{n-k+1}{i} [n-k-i, n-k-i-1]. \end{aligned}$$

Put $j = i + k$ and reverse the order of summation:

$$\begin{aligned} S &= \sum_{j=0}^{n-1} \sum_{k=0}^{j} (-1)^j s(n, n-k) \binom{n-k-1}{j-k} [n-j, n-j-1] \\ &= \sum_{j=0}^{n-1} (-1)^j s(n-1, n-j-1)[n-j, n-j-1] \end{aligned}$$

because of the following identity (which we prove in an appendix):

$$\sum_{k=0}^{j} s(n, n-k) \binom{n-k-1}{j-k} = s(n-1, n-j-1).$$

Now the graph $\Gamma(n-j, m, n-j-1)$ is of the form $\Gamma'(n-1, m-1)$, where $\Gamma' = \Gamma(n, 1)$. So S is given by the same expression for a special $(n-1, m-1)$ graph. By induction, $S = P_{\Gamma_{eq}}(x)$, and we are done.

3 Orbital flow roots

3.1 The orbital flow polynomial

Given a graph Γ, an orientation of the edge set of Γ, and an abelian group A, an *A-flow* on Γ is a function from oriented edges to A with the property that the signed sum of its values on the edges incident with any vertex is 0. The flow is *nowhere-zero* if it never takes the value 0. Tutte [6] showed that the number of nowhere-zero A-flows depends only on the order k of A and not on its structure, and moreover that this number is given by a polynomial

$F_\Gamma(k)$, the *flow polynomial* of Γ. (The flow in any bridge must be zero, so we assume that the graph Γ is bridgeless.)

In [1], the authors showed that the number of orbits of a group G of automorphisms of Γ on nowhere-zero A-flows may depend on the structure of A. More precisely, this number is given by substituting $x_i = \alpha_i(A)$ into a polynomial $OF_{\Gamma,G}(x_0, x_1, x_2, \ldots)$, the *orbital flow polynomial* of Γ and G, where $\alpha_i(A)$ is the number of solutions of $ix = 0$ in A. (Note that $\alpha_0(A) = |A|$ and $\alpha_1(A) = 1$.)

It is shown in [1] that if the variable x_i occurs in the orbital flow polynomial, for $i > 0$, then G must have an element of order i. Thus, if A is an abelian group of order coprime to $|G|$, then the number of G-orbits on nowhere-zero A-flows is given by $OF^1_{\Gamma,G}(k)$, where $k = |A|$ and

$$OF^1_{\Gamma,G}(x) = OF_{\Gamma,G}(x, 1, 1, \ldots).$$

We call OF^1 the *univariate orbital flow polynomial* of Γ and G. (This explains Tutte's observation: if G is the trivial group then every finite abelian group has order coprime to $|G|$.)

3.2 Negative orbital flow roots

Analogous to the result on zero-free intervals for the chromatic polynomial, Wakelin [7] showed that the flow polynomial of a bridgeless connected graph has no roots in $(-\infty, 1)$ (and has sign $(-1)^{m-n+1}$ there, where n and m are the numbers of vertices and edges), and no roots in $(1, \frac{32}{27}]$ (and has sign $(-1)^{n-m+b+1}$ there, where b is the number of blocks).

Modifying our method for orbital chromatic roots, we show:

Theorem 3.1 *The roots of univariate orbital flow polynomials are dense in* $(-\infty, 0]$.

Proof Let Δ consist of two vertices joined by n parallel edges, where n is even. Then the flow polynomial of Δ is known to be $(x-1)((x-1)^{n-1}+1)/x$. So, if Γ consists of the disjoint union of m copies of Δ, and G is the symmetric group of degree m permuting these copies, the number of G-orbits on flows in Γ is obtained by substituting $x = |A|$ into the polynomial

$$OF^1_{\Gamma,G}(x) = \binom{q(x)+m-1}{m}, \text{ where } q(x) = (x-1)((x-1)^{n-1}+1)/x.$$

Hence every solution of $q(x) = -k$, for positive integers k, is an orbital flow root.

Putting $y = 1 - x$, this equation becomes

$$\frac{y^n - 1}{y - 1} = k + 1.$$

The polynomial on the left is clearly positive, and has all derivatives positive, for $y > 0$. So there is a unique real positive root $\alpha(n, k)$, and for fixed n the spacing of the roots decreases as k increases. Clearly $\alpha(n, n-1) = 1$, so we have $\alpha(n, k+1) - \alpha(n, k) < 1/(n-1)$ for $k \geq n$. Hence the values of $\alpha(n, k)$ are dense in $[1, \infty)$, and the values of $1 - \alpha(n, k)$ (which are orbital flow roots) are dense on the negative real line.

We cannot use the second part of the earlier proof to show that the orbital flow roots are dense in $\mathbb{R}$ by "shifting" them up by integers. Indeed, it has been conjectured that a real flow root cannot be larger than 4.

Note that, for n large compared to k, $\alpha(n, k)$ is approximately $1 - 1/(k+1)$, so the corresponding orbital flow root is close to $1/(k+1)$. So, although there are infinitely many orbital flow roots in the interval $[0, 1]$, the only limit points we obtain from this construction are reciprocals of positive integers.

We can modify this example to produce a simple connected planar graph. First, make the graph simple by subdividing each edge. This does not change the number of flows. Then identify the mid-points of one set of corresponding edges from the graphs Δ. Since the net flow on the edges to this mid-point from each copy of Δ must be zero, again no extra flows are produced. See Figure 1 for $n = 4$.

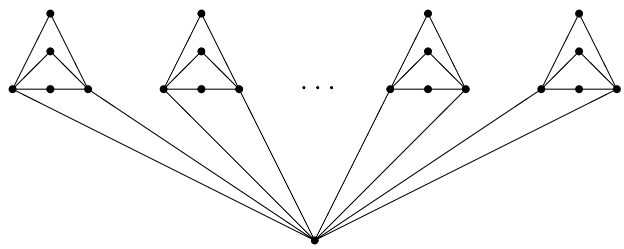

Figure 1: A graph with negative orbital flow roots

Appendix: Proof of an identity

We prove the identity in the second proof of Theorem 2.4.

Multiply both sides by x^{n-j-1} and sum over j from 0 to $n-1$, with the convention that $s(n-1,0)=0$. The right-hand side is the falling factorial $(x)_{n-1}$, by a standard property of the Stirling numbers. On the left, we have

$$\sum_{j=0}^{n-1}\sum_{k=0}^{j} s(n,n-k)x^{n-k-1}\binom{n-k-1}{j-k}(x^{-1})^{j-k}.$$

Now

$$\sum_{j\geq k}\binom{n-k-1}{j-k}(x^{-1})^{j-k}=((x+1)/x)^{n-k-1}$$

by the Binomial Theorem. So the left-hand expression is

$$\sum_{k=0}^{n-1} s(n,n-k)(x+1)^{n-k-1}=(x+1)^{-1}(x+1)_n=(x)_{n-1}.$$

The required identity is obtained by equating coefficients of x^{n-j-1}.

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