## **Problems**

These are problems which have been on my homepage and are now put out to grass. See also permutation group problems.

1. In 1956, Rudin defined a permutation of the integers which maps 3x to 2x, 3x + 1 to 4x + 1, and 3x - 1 to 4x - 1 for all *x*. *Problem*: Determine the cycle structure of this permutation.

I have just learned (December 1998) that this problem is older: it is the "original Collatz problem" from the 1930s (before the famous 3x + 1 problem). A paper by Jeff Lagarias gives details.

2. Let f(k,n) be the number of rooted trees with n leaves, all at level k (that is, distance k from the root), up to isomorphism of rooted trees. Prove that f(k+1,n)/f(k,n) tends to infinity with n, for fixed k. Is it even true that f(k+1,n)/f(k,n) is at least 1 + (n-1)/k?

Solution by Peter Johnson.

Let *r* be the maximum number of edges from a vertex on one level to the next level, in a tree with *n* vertices at level *k*. Then  $r^k \ge n$ , so  $r \ge n^{1/k}$ .

From any tree of height k + 1, we obtain at most k different trees of height k by suppressing one level (replacing the paths of length 2 crossing this level by single edges).

But there is some tree of height k from which at least  $p(n^{1/k})$  trees of height k + 1 can be recovered by introducing a new level, where p is the partition function. (Choose a level where some vertex has at least  $n^{1/k}$  upward edges; if the number of such edges is r, then we can split the r edges in p(r) ways giving non-isomorphic trees. (Treat all the other vertices on this level alike.)

So  $f(k+1,n)/f(k,n) \ge p(n^{1/k})/k$ .

For motivation, see "On the probability of connectedness" in *Discrete Mathematics* **167/168** (1997), 173–185.

<sup>3.</sup> Let *C* be a class of graphs closed under forming disjoint unions and connected components. Let  $c_n$  be the number of connected *n*-vertex graphs in *C*, and  $a_n$  the total number of *n*-vertex graphs in *C* (up to isomorphism). Suppose that c(x) is the generating function of the sequence  $(c_n)$ ; assume that c(x) has non-zero radius of convergence r < 1.

Is it true that, if the ratio  $c_n/a_n$  tends to a positive limit as *n* tends to infinity, then c(r) converges?

Information I have about this is in the paper "On the probability of connectedness" in *Discrete Mathematics* **167/168** (1997), 173–185.

4. The number of strings of n red and blue beads is equal to the number of collections of necklaces of red and blue beads, where rotation of each necklace is permitted and only necklaces with no rotational symmetry are allowed. (The number is  $2^n$  in each case.)

For example, when n = 2, the collections of necklaces are (R)(R), (R)(B), (B)(B), and (RB); (RR) and (BB) are not allowed. The strings are RR, RB, BR, BB.

*Problem*: find a bijective proof of this fact.

Solutions were given by Dima Fon-Der-Flass, Rosemary Bailey, and Robin Chapman. Then Roger Bryant drew my attention to C. Reutenauer, *Free Lie Algebras*, London Math. Soc. Monographs (New Series) 7, Oxford University Press, 1993, which contains this and more.

An account of my secret agenda in asking this question is in the paper "The algebra of an age", in *Model Theory of Groups and Automorphism Groups* (ed. David M. Evans), London Math. Soc. Lecture Notes **244**, Cambridge University Press, 1997, pp. 126–133.

5. Let G be a finite graph, and  $\mathcal{X}(G)$  the class of graphs containing no induced subgraph isomorphic to G. For which G is it true that the probability that a (labelled or unlabelled) graph in  $\mathcal{X}(G)$  with n vertices is connected tends to a limit strictly between zero and one as n tends to infinity?

For example, if *G* is the path of length 3, then the probability of connectedness in *n*-vertex graphs in X(G) is 1/2 for all n > 1.

Solution by Jeannette Janssen:

Use the terms S-sequence and U-sequence for the sequences counted by s(n) and u(n). The left-hand side of the equality counts U-sequences ending in n; removing the last term gives a U-sequence with sum less than n.

<sup>6.</sup> Let s(n) be the number of sequences of elements from the set  $\{1, ..., n\}$  for which each term is at least twice the preceding one, and u(n) the number of such sequences in which each term is greater than the sum of its predecessors. It is known that u(n) - u(n-1) = s(n)/2. *Problem*: Find a bijective proof.

Given an S-sequence, we can add or remove 1 without changing the S-sequence property. So the right-hand side counts S-sequences containing 1.

Now take one of these U-sequences, say  $u_1, u_2, ...$ ; the corresponding S-sequence is  $1, 1 + u_1, 1 + u_1 + u_2, ...$ 

Note: this problem is from my combinatorics textbook, for which a Web page now exists.

7. A triangular prism made of uniform material has cross-section with sides a, b and c (not necessarily equal). It is rolled along a smooth plane so that it rotates about its axis. It gradually loses energy (because of air resistance). What are the probabilities that it lands on each of the three faces (in terms of a, b and c)?

There are obviously some points of detail in the physics to discuss. Dima Fon-Der-Flaass has an argument to show that, if the loss of energy is infinitesimally slow, and the triangle has unequal sides, then the probabilities are either 1/2, 1/4, 1/4 or 1/2, 1/2, 0.

This assumption is not physically realistic. Paul Glendinning found the even more surprising result that there is a collection of non-trivial intervals, arbitrarily close to 0, such that if the dissipation rate d lies in one of these intervals, then the probabilities are 1, 0, 0 for almost all initial conditions; that is, there is one face on which the prism is almost certain to land. See P. Glendinning, "Inaccessible attractors of weakly dissipative systems", *Nonlinearity* **10** (1997), 507–522.

Franco Vivaldi pointed out that, if instead the prism is turned to a random orientation and dropped onto a plane covered with glue, then the probabilities are proportional to the angles subtended by the sides at the centroid of the triangle.

For an infinite oligomorphic group (one with  $F_i$  finite for all *i*), the alternating sum may make sense (i.e. it may converge, or some other summation method may apply), even though the proportion of derangements does not. *Problem*: What does this mean?

For example, in the infinite symmetric group,  $F_i = 1$  for all *i*, and the sum is 1/e, which is just the limiting proportion of derangements in finite symmetric groups.

<sup>8.</sup> It can be shown that, if  $F_i$  is the number of orbits of a finite permutation group *G* on *i*-tuples of distinct elements, then the proportion of elements of *G* which are derangements (have no fixed points) is the alternating sum of the quantities  $F_i/i!$ . (See N. Boston, W. Dabrowski, T. Foguel, P. J. Gies, J. Leavitt, D. T. Ose and D. A. Jackson, "The proportion of fixed-point-free elements of a transitive permutation group", *Communications in Algebra* **21** (1993), 3259–3275.)

A more mysterious example is the group of order preserving permutations of the rational numbers. Here,  $F_i = i!$  for all *i*, and the sum is  $1 - 1 + 1 - 1 \dots = 1/2$  (according to Euler). In what sense are half of the order-preserving permutations derangements?

9. (A. D. Keedwell). Suppose that  $x_1, \ldots, x_m, y_1, \ldots, y_n$  are positive integers such that there exists a bipartite graph with vertex degrees  $x_1, \ldots, x_m$  in one bipartite block and  $y_1, \ldots, y_n$  in the other. (This is equivalent to asserting that the conditions of the Gale-Ryser theorem are satisfied.) Suppose further that all the  $x_i$  and  $y_j$  are greater than 1. Show that there is a bipartite graph having these vertex degrees, which has two proper edge-colourings such that

- for any vertex, the sets of colours appearing on edges at that vertex are the same in both colourings;
- no edge receives the same colour in both colourings.

*Note*: it is not true that any graph with these vertex-degrees has such a pair of colourings. A counterexample is given by the two degree sequences (3,2,2,2) and (3,2,2,2), when the graph consisting of two 4-cycles joined by an edge does not have such a pair of colourings (though the graph consisting of two vertices joined by three disjoint paths of length 3 has such the same degree sequences and does have such colourings.

The conjecture is true if the degrees are all equal, or if every degree is either 2 or 3.

This problem is also in the list of BCC16 problems, available in DVI or PostScript.

Recently, it has been shown by M. Mahdian, E. S. Mahmoodian, A. Saberi, M. R. Salavatipour and R. Tusserkani that this conjecture is a consequence of the celebrated "oriented cycle double cover conjecture" for bridgeless graphs. They have shown its truth in many cases.

10. Let *S* be the symmetric group on the infinite set *X*. Consider the product action of  $S^2$  on  $X^2$ , and let  $a_n$  be the number of orbits on subsets of size *n*.

*Problem*: Find a formula for, or an efficient means of calculating,  $a_n$ . The number an has other combinatorial interpretations:

• It is the number of zero-one matrices with *n* ones, with no zero rows or columns, up to row and column permutation.

- It is the number of bipartite graphs with *n* edges, no isolated vertices, and a distinguished bipartite block, up to isomorphism.
- 11. (a) Do there exist eight subsets of the set  $\{1, \ldots, 10\}$  with the properties
- any two points lie in at most two sets,
- any two sets meet in at least two points?

Answer: yes; I found one by hand:

```
\{1,2,5,8,9\} \\ \{1,2,6,7,9\} \\ \{3,4,5,7,9\} \\ \{3,4,6,8,9\} \\ \{1,3,5,6,10\} \\ \{2,4,5,6,10\} \\ \{1,4,7,8,10\} \\ \{2,3,7,8,10\} \\ \}
```

(b) Do there exist twelve subsets of the set  $\{1, \ldots, 15\}$  with the above properties?

Answer: no (by a combination of case analysis and computer search). Remark: The next interesting open case is eighteen subsets of  $\{1, ..., 21\}$ .

12. Let  $A_n$  denote the *n*th element on the axis of symmetry of Pascal's triangle (the middle binomial coefficient  $\binom{2n}{n}$ ).

A simple argument with generating functions shows that

 $A_0A_n + A_1A_{n-1} + \cdots + A_nA_0 = 4n.$ 

(The generating function  $\sum_{n\geq 0} A_n t^n$  is equal to  $(1-4t)^{-1/2}$ , by the binomial theorem.)

Problem: Find a "counting proof" of this identity.

This problem is due to Alastair King and Andrew Swann. I heard it from Geoff Smith. A solution was found by Omer Egecioglu. Subsequently, Robin Chapman told me that the result is due to Daniel J. Kleitman, "A note on some

subset identites", *Studies in Applied Mathematics* LIV 289-293 (1975). Then an even earlier "proof" was unearthed in W. Feller's classic *An Introduction to Probability Theory and its Applications*, by Alastair King. Here are his comments (with an addition by Robin Chapman):

As observed by Kleitman, the "central binomial convolution identity" amounts to the fact that the number of balanced +1/-1 sequences (random walks) is equal to the number of never balanced ones of the same length.

A simple combinatorial proof of this fact can be extracted from the probability literature:

W. Feller, An Introduction to Probability Theory and its Applications, Wiley, New York, 1950.

The result plays a fundamental role in the analysis of random walks (Main Lemma III.3.1). The proof given in the text is not strictly combinatorial, but Problem III.10.7 asks for a purely combinatorial proof and sketches the construction of a bijection which is attributed to E. Nelson. Confusingly, the sketch appears to construct a bijection with another set of random walks of the same size, namely those which are non-negative. However, a small modification yields the following:

Identify the "initial" segment of a walk as being:

- for a balanced walk, up to the first time it reaches either
  - its minimum value for walks that start -1 or
  - its maximum value for walks that start +1,
- for a never balanced walk, up to the last time it reaches half its final value
  - with a +1 for walks that start with +1 or
  - with a -1 for walks that start with -1.

The bijection reverses the signs and order of the initial segment.

Note: the first and last steps in an initial segment have the same sign. Thus balanced walks that start -1 correspond to never balanced ones that start +1 (and vice versa).

13. This problem is due to Sylvie Corteel, Carla D. Savage, Herbert S. Wilf and Doron Zeilberger, and appeared in the *Journal of Combinatorial Theory*, Series A, 82 (1998), 186–192.

Among the ordered pairs of monic polynomials of degree n over GF(2), exactly half consist of relatively prime polynomials. Find a nice bijection between the relatively prime pairs and the non-relatively-prime pairs.

I have no further information.

14. Two problems on designs and partitions.

1. Is it possible to partition the 4-subsets of a 16-set into 13 sets of 140 subsets, each set isomorphic to the set of planes in the affine space AG(4,2)?

Rudi Mathon informs me that Luc Teirlinck has shown the non-existence of such a partition. I hope to add the reference shortly.

Of course, one can now ask for a partition of the 4-subsets of a 32-set into 29 copies of AG(5,2), and so on. Such a partition would give rise to a new partial geometry.

2. Is there a Steiner system S(3,5,26) (a set of 5-subsets of a 26-set with the property that any 3 points lie in just one of them) having the property that there are no three of the blocks which have pairwise intersections of size 2 but the intersection of all three is empty? Such a design would give rise to a strongly regular graph on 352 vertices and hence to a biplane with 352 points and the same number of blocks.

15. Two problems on random Latin squares

- 1. Call a row of a Latin square even or odd according as it is an even or an odd permutation of the set  $\{1, 2, ..., n\}$ . Prove that the number of even rows of a random Latin square of order *n* is approximately binomial B(n, 1/2). (The best result in this direction is due to R. Häggkvist and J. Janssen, *Discrete Mathematics* **157** (1996), 199-206: the probability that all rows are even is exponentially small.)
- Define a probability measure on permutations of {1,2,...,n} as follows: the probability of a permutation is the proportion of Latin squares with first row (1,2,...,n) in which it occurs as the second row. What can be said about this distribution? (For example, a permutation has non-zero probability if and only if it is a derangement.) In particular, prove that the probability that

a permutation lies in no transitive proper subgroup of the symmetric group except possibly the alternating group tends to 1 as  $n \to \infty$ .

(The analogous result for the uniform distribution was proved by T. Łuczak and L. Pyber, *Combinatorics, Probability and Computing* **2** (1993), 505–512. Their result gives a simple deduction that the group generated by the rows of a random Latin square is almost surely the symmetric group. A solution to this problem would give the same conclusion for a random "normalised" Latin square.)

16. Find all positive integers greater than two which can be written as the sum of two powers of the same prime in three different ways. (I know of only six such integers, none of which has more than three different representations of this form.)

17. It is known that the average number of ways in which a positive integer in the range  $\{1, ..., n\}$  can be written as a sum of consecutive primes tends to the limit  $\log_e 2$  as  $n \to \infty$ . Is it true that the limiting distribution of the *number of representations* of this form is Poisson with parameter  $\log_e 2$ ?

This would imply that the density of the set of numbers with no such representation is 1/2. It would also imply that there exist integers with arbitrarily many such representations. Either of these assertions would be a nice result!

18. A problem on Fibonacci numbers. (I learned this problem from John Conway; see also C. Kimberling, "Numeration systems and fractal sequences", *Acta Arith.* **73** (1995), 103–117.)

Any positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers:

$$n=F_{i_1}+F_{i_2}+\cdots+F_{i_k},$$

where  $i_j \ge i_{j-1} + 2$ . Hence we can define the *Fibonacci successor function* on the natural numbers by taking this expression and "moving each Fibonacci number along one":

$$\sigma(n) = F_{i_1+1} + F_{i_2+1} + \dots + F_{i_k+1}.$$

So, for example, 40 = 34 + 5 + 1; its Fibonacci successor is 55 + 8 + 2 = 65.

Now produce a table in which each row begins with the smallest number which doesn't occur in any earlier row, and all further entries are obtained by applying

the Fibonacci successor function. (So the top row consists of the Fibonacci numbers 1, 2, 3, 5, 8, ...; the next begins 4, 7, 11, ....) It turns out that if we extrapolate each row back two steps using the recurrence relation for the Fibonacci numbers, we obtain the natural numbers 0, 1, 2, 3, ...; this gives a convenient numbering of the rows. Every positive integer occurs exactly once in this table, and every sequence of positive integers satisfying the Fibonacci recurrence relation agrees from some point on with a row of the table.

Now let  $x_n$  denote the fractional part of  $n\tau$ , where  $\tau$  is the golden ratio. Let  $a_n$  and  $b_n$  be the number of earlier terms of this sequence to the left and the right respectively of  $x_n$  (including 0 and 1). The sequence of pairs  $(a_n, b_n)$  begins (1, 1), (1, 2), (3, 1), (2, 3), (1, 5), (5, 2), ...

If we look at the pairs occurring in the Fibonacci-numbered positions, we see  $(1,1), (1,2), (3,1), (1,5), \ldots$ . In other words, one number in the pair is 1, and it "bounces" from side to side as we progress along the sequence. This follows from well-known properties of the continued fraction expansion of  $\tau$ .

*Problems*: If we consider the pairs in the positions numbered by any fixed row in the table, show that there is a "bouncing number" which alternates sides in the pairs. Which numbers occur as "bouncing numbers"?

For example: row number 1 of the table is 4, 7, 11, 18, 29, ...; the pairs in these positions are (2,3), (3,5), (9,3), (3,16), (27,3), ...; so the "bouncing number" is 3. The "bouncing numbers" for rows 0, 1, 2, 3, 4, ... appear to be 1, 3, 2, 5, 8, 5, 9, .... They are predominantly but not exclusively Fibonacci numbers.

Further information here, in DVI or PostScript.

19. Let  $p_1$  be a prime number. For every *n*, let  $p_{n+1}$  be the smallest prime divisor of  $p_1 \cdots p_n + 1$ .

- Is it true that, for every *n*, there is a prime  $p_1$  for which none of the first *n* terms of the sequence is equal to 3?
- Is there a prime  $p_1$  for which no term of the sequence is equal to 3?

(This problem is due to Steve Donkin.)

20. Is the following true?

• If *P* is a 2-group which is not elementary abelian, then some non-identity element of the centre of *P* is a square.

The answer to this question is negative. The smallest counterexamples have order 128, and there are two of them. This was found by Alexander Hulpke and Andreas Caranti. Alexander provided the following example in GAP:

```
gap> g:=SmallGroup(128,36);
gap> z:=Centre(g);
Group([ f6, f7 ])
gap> r:=List(ConjugacyClasses(g),Representative);;
gap> s:=Filtered(r,i->Order(i)>2);;
gap> List(s,Order);
[ 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4 ]
gap> Set(List(s,i->i^2));
[ f4, f5, f4*f7, f5*f6, f3*f4*f5, f3*f4*f5*f6,
    f3*f4*f5*f7, f3*f4*f5*f6*f7 ]
gap> List(last,i->i in z);
[ false, false, false, false, false, false,
    false, false ]
```

If not, is the following weaker statement true?

• If *P* is a 2-group which is not elementary abelian, and *Q* is a core-free subgroup of *P*, then there is an element lying in no conjugate of *Q* which is a square.

Avinoam Mann also suggested the above group as a possible counterexample to the second question. However, Steve Linton and John Murray both checked that there is no core-free subgroup Q for which it is a counterexample. So the question is still open.

2.  $X_i \cap X_j$  is isomorphic to *F* for all distinct *i*, *j*;

3. any edge of the complete graph lies in exactly  $\lambda$  of the graphs  $X_1, \ldots, X_n$ .

<sup>21.</sup> A symmetric graph design, or SGD, with parameters  $(n, X, \lambda, F)$ , (where n and  $\lambda$  are positive integers and X and F are graphs on n vertices), is a set  $\{X_1, \ldots, X_n\}$  of subgraphs of the complete graph on  $\{1, \ldots, n\}$  having the following properties:

<sup>1.</sup>  $X_i$  is isomorphic to X for all *i*;

To avoid degenerate cases, we assume that *X* is neither the complete nor the null graph on *n* vertices. Symmetric graph designs generalise symmetric BIBDs (symmetric 2-designs, the case where  $X = K_k$  and  $F = K_\lambda$ ). In this case, condition 2 follows from conditions 1 and 3.

*Problem*: Does condition 2 follow from 1 and 3 in general? Does 3 follow from 1 and 2? Does 1 follow from 2 and 3? (Assume that  $\lambda > 1$ .)

Dima Fon-Der-Flaass has shown that condition 1 does not follow from 2 and 3. An example for  $\lambda = 2$ : it is easy to find *n* subgraphs of  $K_n$ , each having n - 1 edges, in such a way that every edge belongs to exactly two of them, and every two subgraphs have a single edge in common. The subgraphs can be arbitrary.

22. It is known that, if G is a permutation group in which every non-identity element fixes exactly k points, then either G has a fixed set of size k, or G is one of a finite list of groups (for given k). The problem is to find a good upper bound (in terms of k) for the orders of the groups in this finite list.

For example, when k = 2, the groups are the tetrahedral, octahedral and icosahedral groups, acting on the vertices, edges and faces of the corresponding polyhedra. (Every rotation has an axis, so has just two fixed points in this action.) So the correct bound is 60. (This is a theorem of Iwahori, *J. Fac. Sci., Univ. Tokyo* **11** (1964), 47–64.)

23. Let G be a finite group, a an automorphism of G, and X the semidirect product of G by the group generated by a. Suppose that every element of X not in G has prime order p.

It follows that the centraliser of a in G is a p-group. Let its order be  $p^m$ .

A theorem of Kegel (*Math. Z.* **75** (1961), 373–376) shows that *G* is nilpotent. Then a theorem of Khukhro (*Mat. Zametki* **38** (1985), 652–657) shows that *G* has a normal subgroup whose nilpotency class and index are bounded by functions of *p* and *m*.

Is it true that the nilpotency class of G is bounded by a function of p and m?

It is known that a quaternary relation is a separation relation if and only if its restriction to every 5-element set is a separation relation.

<sup>24.</sup> A separation relation on a set is the quaternary relation induced by a circular order on the set by the rule that S(a, b, c, d) holds if a and b separate c and d. (This holds for exactly one of the three partitions of the four points a, b, c, d into two pairs.)

Let *P* be an inversive plane, *C* a circle of *P*. Define a relation *S* on *C* by the rule that S(a,b,c,d) holds if every circle through *a* and *b* meets every circle through *c* and *d*.

*Problem*: What does it mean for *P* if, for every circle *C* of *P*, the above relation *S* is a separation relation?

Peter J. Cameron March 28, 2000