Root systems and optimal block designs

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Abstract

Motivated by a question of C.-S. Cheng on optimal block designs, this paper describes the symmetric matrices with entries 0, $+1$ and -1, zero diagonal, least eigenvalue strictly greater than -2, and constant row sum. I also describe briefly the motivation for the question.

Let A be a symmetric $n \times n$ matrix with entries 0, $+1$ and -1, with zero diagonal and constant row sums, having least eigenvalue greater than -2. The aim of this paper is to describe such matrices.

Of course, we may assume that the matrix is "connected", that is, not permutation-equivalent to one of the form $\begin{pmatrix} B & O \\ O & C \end{pmatrix}$. We also note that the constant row sum c is an eigenvalue, and so $c \geq -1$.

Such a matrix is represented by a set of vectors in a spherical root system, and hence (apart from finitely many examples represented in the exceptional root systems E_6, E_7 and E_8) by either a tree with oriented edges, or a unicyclic graph with edges either signed or oriented. We give a test for recognising when such a graph represents a matrix satisying the conditions of the question. There are many examples. All matrices occurring in the exceptional root systems are determined.

1 Least eigenvalue -1

As a warmup, I consider the case where the least eigenvalue is -1.

Let A be such a matrix. Then $A+I$ is positive semi-definite, and so is the Gram matrix of inner products of n vectors $v_1,\ldots,v_n$ in Euclidean space. Thus $||v_i|| = 1$ and $v_i \cdot v_j \in \{0,+1,-1\}$ for all i,j. Clearly such vectors consist of an orthonormal set of vectors and their negatives, with each vector possibly repeated. Connectedness implies that there is just one vector v in the set. So, if we only use the vector v, all inner products are $+1$, and we have $A = J-I$, where J is the all-1 matrix; if we use both v and $-v$, then $A = \begin{pmatrix} J-I & -J \\ -J & J-I \end{pmatrix}$.

Note that the first type has constant row sum $n-1$; the second type has constant row sum -1 if and only if all the blocks are square, that is, v and $-v$ occur equally often in the vector representation. Each matrix has the property that the entry 0 doesn't occur. We call these the *trivial* examples.

2 Root systems

For the initial analysis we ignore the "constant row sum" condition, and also for a while we only assume that the least eigenvalue is not smaller than -2.

Let A be such a matrix. Then $A+2I$ is positive semi-definite, so is the matrix of inner products of a set of vectors $v_1,\ldots,v_n$, where $||v_i|| = \sqrt{2}$ and $v_i \cdot v_j \in \{0,+1,-1\}$ for all $i \neq j$. Also, the matrix A can be assumed to be "connected", so there is no orthogonal decomposition of the Euclidean space so that the vectors all lie in the summands. (This means that the graph with vertex set $v_1,\ldots,v_n$, with two vertices adjacent if they are not orthogonal, is connected.) We can assume that they span the space.

According to the results of [1], the vectors form a subset of a root system of type A_n, D_n, E_6, E_7 or E_8. Here is a brief account.

A *root system* is a finite set of non-zero vectors in Euclidean space $\mathbb{R}^n$, closed under reflection in the hyperplane perpendicular to each of its elements. It is *indecomposable* if it is not contained in the union of two non-zero perpendicular subspaces. Indecomposable root systems arose in the classification of simple Lie algebras; they were determined by Cartan and Killing.

We are interested in the root systems with all roots of the same length. Here is the list in this case; there are two infinite families and three sporadic examples. Let $e_1,\ldots,e_n$ form an orthonormal basis for $\mathbb{R}^n$.

A_n: The vectors of A_n are all $e_i - e_j$, for $1 \leq i,j \leq n+1$, $i \neq j$. (These vectors lie in a hyperplane of $\mathbb{R}^{n+1}$, and so span a space of dimension n.)

D_n: The vectors of D_n are all $\pm e_i \pm e_j$, for $1 \le i < j \le n$.

E_n: These are three specific sets in $\mathbb{R}^n$, for $n = 6, 7, 8$. Several descriptions of them can be found in [1].

So our matrix is represented by a subset of one of these root systems, where we choose at most one out of each pair $v, -v$.

Now let us make the assumption that the least eigenvalue of A is strictly greater than -2. Then $A + 2I$ is positive definite, so the representing vectors $v_1, \ldots, v_n$ are linearly independent, and form a basis for $\mathbb{R}^n$.

Thus, let us say that an $n \times n$ matrix A is *admissible* if

- A is real symmetric, having entries 0, $+1$, and -1 only;
- the diagonal entries are all 0;
- A is connected;
- the smallest eigenvalue of A is greater than -2.

Our problem is to determine the admissible matrices with constant row sum. This is equivalent to choosing a connected subset of a root system whose vectors form a vector space basis for the ambient space, and such that the Gram matrix of the subset has constant row sums.

3 Determinant

In this section, we show:

Proposition 3.1 *Let A be an admissible $n \times n$ matrix. Then* $\det(2I + A) = n + 1$ *or* 4*, except possibly if* $n = 6$*,* 7 *or* 8*, when* $\det(2I + A)$ *may be* 3*,* 2 *or* 1 *respectively.*

This follows from the following result. The *root lattice* associated with a root system is the integer span of the root system. (The term "lattice" here means "discrete spanning subgroup of $\mathbb{R}^n$".) We refer to Humphreys [3] for more information about root systems and root lattices.

Proposition 3.2 *Let S be a connected subset of a root system R which forms a vector space basis for the ambient space. Then S is an integer basis for the corresponding root lattice, except possibly if $R = E_7$ or $R = E_8$, in which case S may be an integer basis for the A_7, A_8 or D_8 root lattice.*

Proof The integer span $L = \langle S \rangle_{\mathbb{Z}}$ is a sublattice of the root lattice. Now $L \cap R$ is a root system. (This requires that, for two vectors v, w in this set with $v \cdot w = \epsilon = \pm 1$, we have $v - \epsilon w \in L \cap R$; this holds since R is closed under this operation by definition and L is a lattice.) Moreover, $L \cap R \supseteq S$, so $L \cap R$ is connected. Thus, $L \cap R$ is a root system of type A_n, D_n or E_n, contained in the given root system. Now the only inclusions among these root systems are $A_7 \subseteq E_7$, $A_8 \subseteq E_8$ and $D_8 \subseteq E_8$.

Now the proof of Proposition 3.1 proceeds as follows. Let L be an integral lattice in $\mathbb{R}^n$ (this means that the inner product of any two of its vectors is an integer). The *dual lattice* L^* consists of all vectors $v \in \mathbb{R}^n$ such that $v \cdot w \in \mathbb{Z}$ for all $w \in L$. Clearly L^* is a lattice containing L, and the index $|L^*/L|$ is finite (this number is the *connection number* of the lattice). If S is any integral basis for L, then the determinant of the Gram matrix of S is equal to the connection number of L.

The connection numbers of A_n, D_n, E_6, E_7 and E_8 are $n+1$, 4, 3, 2 and 1 respectively. This finishes the proof.

Hence, if we determine the admissible matrices, we can decide which root system contains each matrix simply by calculating its determinant. The result also restricts the possible row sum s, since $s + 2$ must divide the determinant.

4 The case A_n

We have a basis $\{v_1, \ldots, v_n\}$ for the root system A_n. We can represent it as a directed graph on $n+1$ vertices as follows: if $v_i = e_j - e_k$, we represent v_i as a directed edge from e_k to e_j. This graph contains no circuits, since the sum (with appropriate signs) of the vectors corresponding to the edges in a circuit is zero. Since the vectors form a basis, the graph is a tree.

Now, given a directed tree with n edges, the matrix A is constructed as follows. Entries on the diagonal, or corresponding to a pair of edges with no common vertex, are zero. The entry corresponding to a pair of edges meeting at a vertex is -1 if the edges are "head-to-tail" there, or $+1$ if they are "head-to-head" or "tail-to-tail".

The constant row sum condition means that, for any edge, if we calculate the entries as above for all edges meeting the given edge at a vertex and sum them, the result is a constant c.

For any vertex v of the tree, let $d(v)$ be the degree of v, and $s(v)$ the "signed degree" (the number of incoming edges minus the number of outgoing edges). Then the sum of the row of A corresponding to the directed edge $v \to w$ from v to w is

$$(-1)(s(v)+1) + (+1)(s(w)-1) = s(w) - s(v) - 2.$$

So we have the following result:

Theorem 4.1 *An admissible matrix having row sums c arises from an oriented tree T if and only if $s(w) - s(v) = c + 2$ for every directed edge $v \to w$ of T. Reversing the orientation of every edge does not change the matrix.*

We note a couple of consequences.

Corollary 4.2 *Let T be an oriented tree satisying the above conditions. Suppose, without loss of generality, that there is an edge directed out of a leaf x. Then all values of $s(v)$ are congruent to $-1 \bmod c+2$, and if the vertices are arranged on levels corresponding to the values of $s(v)$, then each oriented edge goes from a level to the next level above.*

Proof We lose no generality because we may reverse all orientations. If x is as in the statement, then $s(x) = -1$, and the theorem together with the connectedness of T shows that all values of $s(v)$ are congruent modulo $c+2$.

Corollary 4.3 *If $c \notin \{-1, 0\}$, then there cannot be both a leaf with an outgoing edge and a leaf with an incoming edge.*

Proof If x and y are such leaves, then $s(x) = -1$ and $s(y) = 1$.

Corollary 4.4 *If c is even, then all vertices of the tree have odd degree. If c is odd, then the parity of the degrees is even in one bipartite block and odd in the other; in particular, all leaves lie in the same bipartite block.*

Proof For any edge $v \to w$, we have $s(w) - s(v) = c + 2$, and $s(v) \equiv d(v) \bmod 2$, so $d(v) + d(w) \equiv c \bmod 2$.

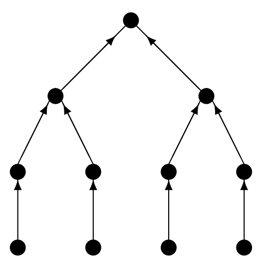
If c is even, all degrees have the same parity; since there exist leaves, the parity is odd. If c is odd, the two ends of an edge have degrees of opposite parity, and the conclusion of the lemma follows.

Corollary 4.5 *Suppose that there is a vertex v which is on an edge towards a leaf and an edge from another leaf. Then the row sum is -1, and the degree of v is even, with half the edges entering v and half of them leaving.*

Proof Let $v \to x$ and $y \to v$ be the two given edges, where x and y are Then $s(x) = +1$ and $s(y) = -1$, so the row sums corresponding to (v, x) and (y, v) are respectively $-s(v) - 1$ and $s(v) - 1$. These are equal, so $s(v) = 0$ and the row sum is -1.

Now the examples $J - I$ and $\begin{pmatrix} J-I & -J \\ -J & J-I \end{pmatrix}$ are realized by stars. In the first case, direct all the edges in to the centre; in the second, let the number of edges be even, and direct half of them in and half of them out.

There are many examples of oriented trees satisfying our conditions. Here is the smallest one with row sums -1 which is not a star.

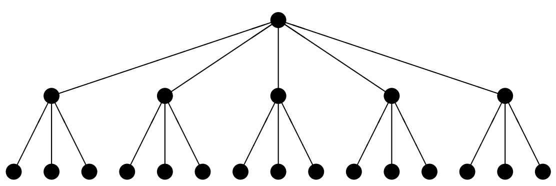


Numbering the edges from left to right and from bottom to top gives the admissible matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 \\ - & 0 & 0 & 0 & 0 & + & 0 & 0 & - & 0 \\ 0 & - & 0 & 0 & + & 0 & 0 & 0 & - & 0 \\ 0 & 0 & - & 0 & 0 & 0 & 0 & + & 0 & - \\ 0 & 0 & 0 & - & 0 & 0 & + & 0 & 0 & - \\ 0 & 0 & 0 & 0 & - & - & 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 & 0 & 0 & - & - & + & 0 \end{pmatrix}$$

whose least eigenvalue, according to Maple, is about -1.860805854.

Examples with other values of c are easily produced. For example, take a star with $2c+3$ leaves all directed outwards. Identify each leaf with the centre of a star with $c+2$ leaves all directed outwards. This gives a matrix of order $(c+3)(2c+3)$ with row sums c. The case $s=0$ gives a matrix of order 9, the smallest non-trivial matrix represented in A_n. The picture shows the case $c=1$. Directions are not shown: all edges are oriented upwards.

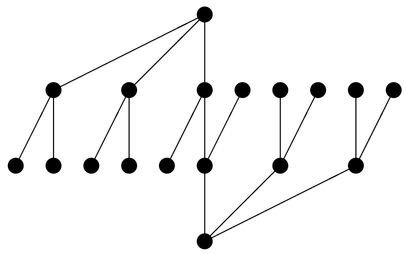


One can give constructions for new examples from old. For example, the following is obviously true, and gives a construction for infinitely many examples. This example only works in the case where row sums are -1.

Proposition 4.6 *Let T_1 and T_2 be oriented trees giving rise to matrices satisfying the conditions of the problem, and for $i=1,2$, let v_i be a vertex of T_i satisfying $s(v_i)=0$. Then the tree formed from the disjoint union of T_1 and T_2 by identifying v_1 and v_2 also satisfies the conditions.*

Other recursive constructions are also possible. Rather than formulate general conditions, we give an example of two trees glued along an edge.

This also shows that the case where row sums are zero and there are initial and terminal leaves can occur. Again, edges are oriented upwards.



5 The case D_n

In this case, our vectors are of the form $\pm e_i \pm e_j$ for $i \neq j$. Again the set can be represented by a graph, this time with n vertices and n edges. The graph is connected, and so is a unicyclic graph (consisting of a cycle with trees attached at some of its vertices).

There are several kinds of edges. A vector $e_i - e_j$ can be represented by a directed edge from j to i. A vector $e_i + e_j$ can be represented by an undirected edge carrying a $+$ sign (i.e. positive at both ends), while a vector $-e_i - e_j$ can be regarded as an undirected edge carrying a $-$ sign. We define $s(v)$ similarly to before: it is the number of positive undirected or incoming directed edges at v, minus the number of negative undirected or outgoing edges at v. Now we have to take a little care to ensure that the vectors are linearly independent. The result is as follows.

Theorem 5.1 *Let G be a unicyclic graph with each edge signed or directed. Then G corresponds to a matrix satisfying the conditions and having row sums c if and only if the following hold:*

(a) the cycle of G contains an odd number of undirected edges;

(b) if $v \to w$, then $s(w) - s(v) = c + 2$; if $\{v, w\}$ is undirected with sign ϵ, then $\epsilon(s(v) + s(w)) = c + 2$.

Proof Condition (a) guarantees that the edges in the cycle correspond to linearly independent vectors. It is clear that there are no other possible dependencies. Condition (b) is, as in the previous case, the translation of the “row sum c” condition.

This time, however, we have the freedom of changing the signs of the basis vectors arbitrarily. For example, changing e_i to $-e_i$ will change a directed edge leaving i to an undirected edge with sign $+$ (and *vice versa*), and a directed edge entering i to an undirected edge with sign $-$ (and *vice versa*). We can exploit this freedom as follows:

Lemma 5.2 *A connected set of vectors forming a basis for D_n can be represented by a unicyclic graph in which all edges except perhaps one are undirected. If the unique cycle has odd length, then all edges are undirected; if it has even length, then there is a directed edge contained in the cycle.*

Proof Temporarily remove an edge from the cycle to leave a tree. Working from a leaf of the tree, change signs of basis vectors so that each edge of the tree is undirected. Now, since the vectors in the cycle are linearly independent, it is easy to see that the remaining edge is undirected or directed according as the cycle has odd or even length.

Note that examples do exist:

Proposition 5.3 *Suppose that the graph is a cycle of length n.*

(a) If n is odd, then the row sums are $+2$, and all the signs can be taken to be $+$. If $n = 2r + 1$, the eigenvalues of A are $2\cos(2j\pi/(2r + 1))$ for $j = 0, \ldots, 2r$, the smallest occurring when $j = r$.

(b) If n is even, then $n \equiv 2 \bmod 4$, the row sums are 0, and the undirected edges have signs $(++--++\cdots--+)$, while the directed edge points from vertex 1 to vertex n. If $n = 4r + 2$, the eigenvalues of A are $\pm 2\sin(2j\pi/(2r + 1))$ for $j = 0, \ldots, 2r$.

Proof (a) Suppose that n is odd, so that all the edges are undirected. Clearly the row sums are either 0 or 2. If they are 0, then the two edges meeting a given edge have different signs, so the signs are $(++--++\cdots)$, which is not possible for odd n. So the row sums are 2, and all edges have the same sign, which we can assume is $+$. Now the matrix A is the adjacency matrix of the n-cycle, whose least eigenvalue is as claimed. Note that these matrices do not contain the entry -1. For $n = 5$, the smallest eigenvalue is $-(\sqrt{5} + 1)/2 = -1.618033988$.

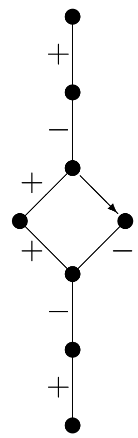
(b) Suppose that n is even, so that there is a single directed edge, which we may suppose is between vertices 1 and n. If the row sums are 2, then all undirected edges have the same sign, which implies that the directed edge has row sum 0, a contradiction. So the row sums are 0. Now, as in the preceding paragraph, each undirected edge apart from $\{1,2\}$ and $\{n-1,n\}$ has neighbours of opposite sign, while the neighbours of the directed edge have the same sign. This implies that $n \equiv 2 \bmod 4$ and the signs are as claimed.

Now a bit of rearranging shows that A can be written in the form

$$\begin{pmatrix} O & C - C^\top \\ C^\top - C & O \end{pmatrix},$$

where C is the matrix of a directed $(2r+1)$-cycle. In this form it is not hard to calculate the eigenvalues. For $n = 6$, the smallest eigenvalue is $-\sqrt{3} = -1.732050808$. In general, the smallest occurs when j is nearest to $(2r+1)/4$ or $3(2r+1)/4$.

Other examples than cycles can occur. Here is one, with least eigenvalue -1.813606504.



$$\begin{pmatrix}
0 & - & 0 & 0 & 0 & 0 & 0 & 0 \\
- & 0 & - & + & 0 & 0 & 0 & 0 \\
0 & - & 0 & - & + & 0 & 0 & 0 \\
0 & + & - & 0 & 0 & - & 0 & 0 \\
0 & 0 & + & 0 & 0 & - & - & 0 \\
0 & 0 & 0 & - & - & 0 & + & 0 \\
0 & 0 & 0 & 0 & - & + & 0 & - \\
0 & 0 & 0 & 0 & 0 & 0 & - & 0
\end{pmatrix}$$

Some analogues of results in the A_n case hold. In particular, if the row sum s is even, then all vertex degrees have the same parity (which is even if the graph consists of a single cycle, as in the lemma, and is odd otherwise, since leaves will exist); and if s is odd, then the graph is bipartite (so the unique cycle has even length), and vertices in different bipartite blocks have degrees of opposite parity.

6 The case E_n, for $n = 6, 7, 8$

Now we determine all admissible matrices with constant row sum of order at most 8; this includes all matrices which generate exceptional root lattices.

The search strategy is as follows. If A is admissible with constant row sums, then the unsigned version of A is the adjacency matrix of a connected graph in which all vertex degrees have the same parity. Odd parity can only arise in the case when n is even, in which case the complement of the graph has all vertices of even parity. So we begin with a list of the Eulerian graphs (with even parity), include also their complements if n is even, and then select just the connected graphs from the list. The Eulerian graphs on small numbers of vertices are available from Brendan McKay's web page [4].

Now we take each such graph, sign the edges in all possible ways, and test to see whether the resulting matrix is admissible and has constant row sums. This is done with a GAP program [2]. We also test isomorphism using the GAP package DESIGN [6].

As explained earlier, we determine the root system for any such matrix by calculating its determinant. We find that the number of such matrices in the exceptional root systems E_n are 2, 4, 12 for $n = 6, 7, 8$ respectively. The output also includes the matrices in A_n and D_n, as a check on our earlier results. The matrices which generate the exceptional root systems are listed in the Appendix.

Here are counts of the admissible matrices of order n with constant row sums for $n \leq 8$, classified by the type of root lattice they generate. The trivial types (in A_n) are $J - I$ and (for n even) $\begin{pmatrix} J-I & -J \\ -J & J-I \end{pmatrix}$. We see that this accounts for all matrices in A_n for $n \leq 8$; the smallest non-trivial example has $n = 9$. The matrices in D_n are the cycles for $n = 5, 6, 7$ and three including the example given in the preceding section for $n = 8$. The matrices are given in the Appendix.

n	3	4	5	6	7	8
A_n	1	2	1	2	1	2
D_n		0	1	1	1	3
E_n				2	4	12

An alternative search strategy would be to examine all the bases for the root systems E_6, E_7 and E_8. Bray (personal communication) has done this; his results agree with those reported in this paper.

7 Which entries may be missing?

We allow the entries 0, $+1$ and -1 in the matrix A. What happens if not all of these entries occur?

In this section I will ignore the exceptional root systems of type E_6, E_7 and E_8. In principle they can contribute at most a finite number of counterexamples to the assertion of the following result; inspection shows that there are none.

Proposition 7.1 *Suppose that A satisfies the usual conditions, and not all values occur. Then $A = J - I$, or $A = \begin{pmatrix} J-I & -J \\ -J & J-I \end{pmatrix}$ with square blocks, or A is the adjacency matrix of an odd cycle.*

Proof We divide into three cases according to the missing value.

Case -1 does not occur: In this case A is the usual adjacency matrix of the line graph of the graph formed by the representing vectors in the root system A_n or D_n. (Directions can be ignored.)

Now, if the line graph of a graph G is regular, then either G is regular, or G is semiregular bipartite (this means that the degrees are constant within each bipartite block).

Subcase G is a tree: One bipartite block must consist of all the leaves, and so every edge goes from a leaf to a non-leaf. Thus G is a star, and $A = J - I$.

Subcase G is unicyclic: If there are leaves, then they form a bipartite block, which is clearly impossible. So our graph G is a cycle. But the line graph of an even cycle has eigenvalue -2. So G is an odd cycle and is isomorphic to its line graph.

Case 0 does not occur: In this case, the graph has the property that any two edges meet. So it must be a star (in case A_n) or a 3-cycle (in case D_n). We get the "trivial" examples of the first section.

Case $+1$ does not occur: In this case, the inner products of the basis vectors are all non-positive. For type A_n, an easy argument shows that the bases are

$$e_1 - e_2, e_2 - e_3, \ldots, e_n - e_{n+1},$$

which gives a Gram matrix with constant row sums only in the trivial cases $n = 2$ and $n = 3$. In the case D_n, we cannot have a cycle of length greater than 2, since each basis vector would occur twice with opposite signs in the representing vectors whose sum would be zero. So, without loss, we have $e_{n-1} \pm e_n$, from which it is easy to see that the basis is

$$e_1 - e_2, e_2 - e_3, \ldots, e_{n-2} - e_{n-1}, e_{n-1} - e_n, e_{n-1} + e_n,$$

which never gives constant row sum for $n > 2$.

Note that the bases here are the standard bases for the root systems, as used in the theory of Lie algebras etc.

8 Optimal block designs

I conclude by describing the background in optimal design theory of the question of Cheng which motivates this research. For further details, see [5].

A *block design* here means a 1-design, or binary proper equireplicate block design. Thus, there are v points; each block is a set of k points; and each point is contained in r blocks.

The *incidence matrix* N of a block design D is the $v \times b$ matrix (where b is the number of blocks) with i, j entry 1 if the ith point is contained in the jth block, 0 otherwise. The *concurrence matrix* $\Lambda = NN^\top$ is the $v \times v$ matrix whose i, j entry is the number of blocks containing the ith and jth points. The *information matrix* L is given by $L = rI - \Lambda/k$. The information matrix has a “trivial” eigenvalue 0, corresponding to the all-1 eigenvector.

Several notions of optimality of block designs have been proposed. A block design D is *A-optimal* (in the class of all block designs with given v, k, r) if it maximizes the harmonic mean of the non-trivial eigenvalues; it is *D-optimal* if it maximizes the geometric mean of the non-trivial eigenvalues; and it is *E-optimal* if it maximizes the smallest non-trivial eigenvalue. (We stress that the letters A, D, E here have no relation to the names of the root systems).

If a balanced design (a 2-design) exists, then it is optimal in all three senses. But if the parameters are such that no balanced design exists, the question of optimality is more subtle, and there may be no design which is optimal on all criteria. Cheng's question was motivated by a search for E-optimal designs. The idea is that, for a balanced design, we have $\Lambda = (r-\lambda)I + \lambda J$, where J is the all-1 matrix; so it is reasonable to search for designs whose concurrence matrix is almost of this form, that is, of the form $(r-t)I + tJ - A$, where A is a symmetric integer matrix with small entries (say, $-1, 0, 1$) and constant row sums. To maximize the least eigenvalue of L, we should make the least eigenvalue of A as large as possible (say, greater than -2). This gives precisely the problem addressed in this paper, with $v = n$.

Two questions remain. First, given a matrix A of order v, can we find a block design with concurrence matrix $(r-t)I + tJ - A$? Second, is such a design in fact E-optimal?

The first question can be readily answered by the `DESIGN` software. Having chosen the block size k, we first choose r and t such that

$$\begin{aligned} t(v-1) - c &= r(k-1), \\ k &\mid vr, \end{aligned}$$

where c is the row sum of A. Then we can calculate Λ, and use `DESIGN` to find a block design with the given v, k, r and with concurrence matrix Λ. For example, if A is one of the matrices in the root system E_6 (see the appendix), and $k = 3$, then we have $c = -1$, so $2r = 5t + 1$. The smallest solution has $t = 1$, $r = 3$; we find that there are no solutions for either matrix. However, for $t = 3$, $r = 8$, the second matrix gives us a unique design, with block set

$$\{123, 125, 125, 134, 136, 136, 146, 156, 234, 245, 246, 246, 256, 345, 345, 356\}.$$

The twelve matrices in E_8 all have $c = -1$. For $k = 3$, the smallest feasible values are $r = 18$, $t = 5$, where designs exist for each of the twelve matrices. For $k = 4$, the smallest feasible values $r = 5$, $t = 2$ cannot be realised, but for the next values $r = 12$, $t = 5$, once again designs exist for each of the twelve matrices.

I have not investigated the second question.

9 Appendix: Matrices of order at most 8

We list here the admissible matrices with constant row sums having order at most 8.

9.1 Matrices in A_n

We have only the trivial matrices $J-I$ and (for n even) $\begin{pmatrix} J-I & -J \\ -J & J-I \end{pmatrix}$.

9.2 Matrices in D_n, for $n \geq 4$

For $n \leq 7$ we have the odd cycles C_5 and C_7 with all signs $+$, and, for $n=6$, the cycle signed as described in Proposition 5.3(b), giving the matrix $\begin{pmatrix} O & A \\ A^\top & O \end{pmatrix}$, where

$$A = \begin{pmatrix} 0 & - & + \\ + & 0 & - \\ - & + & 0 \end{pmatrix}.$$

For $n=8$, there are three matrices:

$$\begin{pmatrix} 0 & + & 0 & 0 & - & 0 & 0 & 0 \\ + & 0 & 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & + & - & 0 \\ 0 & 0 & 0 & 0 & 0 & + & 0 & - \\ - & - & 0 & 0 & 0 & 0 & + & + \\ 0 & 0 & + & + & 0 & 0 & - & - \\ 0 & 0 & - & 0 & + & - & 0 & + \\ 0 & 0 & 0 & - & + & - & + & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & + & 0 & 0 & - & - \\ 0 & 0 & + & - & 0 & - & 0 & 0 \\ 0 & + & 0 & - & - & 0 & 0 & 0 \\ + & - & - & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & + & - & 0 & - \\ 0 & 0 & - & + & 0 & 0 & - & 0 \\ 0 & - & 0 & - & 0 & + & 0 & 0 \\ 0 & + & - & 0 & - & 0 & 0 & 0 \\ + & 0 & 0 & - & 0 & - & 0 & 0 \\ - & 0 & + & 0 & - & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

9.3 Matrices in E_6

$$\begin{pmatrix} 0 & - & + & + & - & - \\ - & 0 & - & - & + & + \\ + & - & 0 & 0 & 0 & - \\ + & - & 0 & 0 & - & 0 \\ - & + & 0 & - & 0 & 0 \\ - & + & - & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & - & + & 0 & - \\ 0 & 0 & + & - & - & 0 \\ - & + & 0 & - & 0 & 0 \\ + & - & - & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

9.4 Matrices in E_7

$$\begin{pmatrix} 0 & 0 & 0 & - & 0 & 0 & + \\ 0 & 0 & 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & 0 & 0 & - & + \\ - & 0 & 0 & 0 & + & + & - \\ 0 & - & 0 & + & 0 & + & - \\ 0 & 0 & - & + & + & 0 & - \\ + & + & + & - & - & - & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & + & 0 & 0 & 0 & 0 & - \\ + & 0 & 0 & 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & 0 & 0 & - & + \\ 0 & 0 & - & 0 & 0 & + & 0 \\ 0 & 0 & 0 & - & + & 0 & 0 \\ - & - & + & + & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & - & 0 & 0 & + & 0 & 0 \\ - & 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & - & + \\ 0 & 0 & 0 & 0 & - & 0 & + \\ + & 0 & 0 & - & 0 & + & - \\ 0 & + & - & 0 & + & 0 & - \\ 0 & 0 & + & + & - & - & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & - & - & + & + & 0 & 0 \\ - & 0 & + & - & 0 & + & 0 \\ - & + & 0 & 0 & - & 0 & + \\ + & - & 0 & 0 & 0 & - & + \\ + & 0 & - & 0 & 0 & + & - \\ 0 & + & 0 & - & + & 0 & - \\ 0 & 0 & + & + & - & - & 0 \end{pmatrix}$$

9.5 Matrices in E_8

$$\begin{pmatrix} 0 & - & + & + & - & - & + & - \\ - & 0 & - & - & + & + & - & + \\ + & - & 0 & + & - & - & 0 & 0 \\ + & - & + & 0 & - & - & 0 & 0 \\ - & + & - & - & 0 & + & 0 & 0 \\ - & + & - & - & + & 0 & 0 & 0 \\ + & - & 0 & 0 & 0 & 0 & 0 & - \\ - & + & 0 & 0 & 0 & 0 & - & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & + & - & - & + & 0 & - & 0 \\ + & 0 & - & - & + & - & 0 & 0 \\ - & - & 0 & + & - & + & 0 & 0 \\ - & - & + & 0 & - & 0 & + & 0 \\ + & + & - & - & 0 & 0 & 0 & - \\ 0 & - & + & 0 & 0 & 0 & - & 0 \\ - & 0 & 0 & + & 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & - & + & - & - & + & 0 & 0 \\ - & 0 & - & + & + & 0 & 0 & - \\ + & - & 0 & - & 0 & 0 & 0 & 0 \\ - & + & - & 0 & 0 & 0 & 0 & 0 \\ - & + & 0 & 0 & 0 & - & 0 & 0 \\ + & 0 & 0 & 0 & - & 0 & - & 0 \\ 0 & 0 & 0 & 0 & 0 & - & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & - & - & + & + & - & + & - \\ - & 0 & + & - & 0 & 0 & - & + \\ - & + & 0 & - & - & + & 0 & 0 \\ + & - & - & 0 & 0 & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 & - & 0 & 0 \\ - & 0 & + & 0 & - & 0 & 0 & 0 \\ + & - & 0 & 0 & 0 & 0 & 0 & - \\ - & + & 0 & 0 & 0 & 0 & - & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & + & - & - & + & - & 0 & 0 \\ + & 0 & - & - & 0 & 0 & 0 & 0 \\ - & - & 0 & + & 0 & 0 & 0 & 0 \\ - & - & + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 & - & - & 0 \\ - & 0 & 0 & 0 & - & 0 & + & 0 \\ 0 & 0 & 0 & 0 & - & + & 0 & - \\ 0 & 0 & 0 & 0 & 0 & 0 & - & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & - & - & + & 0 & 0 & + & - \\ - & 0 & + & - & 0 & - & 0 & + \\ - & + & 0 & - & + & 0 & - & 0 \\ + & - & - & 0 & - & + & 0 & 0 \\ 0 & 0 & + & - & 0 & 0 & 0 & - \\ 0 & - & 0 & + & 0 & 0 & - & 0 \\ + & 0 & - & 0 & 0 & - & 0 & 0 \\ - & + & 0 & 0 & - & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & - & - & + & + & 0 & 0 & - \\ - & 0 & + & 0 & - & 0 & 0 & 0 \\ - & + & 0 & - & 0 & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 & - & 0 & 0 \\ + & - & 0 & 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & - & - & 0 & - & + & + & 0 \\ - & 0 & + & + & 0 & - & - & 0 \\ - & + & 0 & 0 & + & 0 & - & - \\ 0 & + & 0 & 0 & - & - & 0 & 0 \\ - & 0 & + & - & 0 & 0 & 0 & 0 \\ + & - & 0 & - & 0 & 0 & 0 & 0 \\ + & - & - & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & - & 0 & 0 & - & - & + & + \\ - & 0 & + & + & 0 & 0 & - & - \\ 0 & + & 0 & 0 & 0 & - & 0 & - \\ 0 & + & 0 & 0 & - & 0 & - & 0 \\ - & 0 & 0 & - & 0 & + & 0 & 0 \\ - & 0 & - & 0 & + & 0 & 0 & 0 \\ + & - & 0 & - & 0 & 0 & 0 & 0 \\ + & - & - & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & + & - & + & - & 0 & - \\ 0 & 0 & + & - & - & + & - & 0 \\ + & + & 0 & - & 0 & 0 & - & - \\ - & - & - & 0 & 0 & 0 & + & + \\ + & - & 0 & 0 & 0 & - & 0 & 0 \\ - & + & 0 & 0 & - & 0 & 0 & 0 \\ 0 & - & - & + & 0 & 0 & 0 & 0 \\ - & 0 & - & + & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$
\begin{pmatrix}
0 & 0 & + & - & - & - & + & 0\\
0 & 0 & - & + & - & 0 & 0 & 0\\
+ & - & 0 & - & 0 & 0 & 0 & 0\\
- & + & - & 0 & 0 & 0 & 0 & 0\\
- & - & 0 & 0 & 0 & + & 0 & 0\\
- & 0 & 0 & 0 & + & 0 & - & 0\\
+ & 0 & 0 & 0 & 0 & - & 0 & -\\
0 & 0 & 0 & 0 & 0 & 0 & - & 0
\end{pmatrix}
\qquad
\begin{pmatrix}
0 & 0 & 0 & - & + & + & - & -\\
0 & 0 & - & 0 & + & - & + & -\\
0 & - & 0 & 0 & - & + & - & +\\
- & 0 & 0 & 0 & - & - & + & +\\
+ & + & - & - & 0 & 0 & 0 & -\\
+ & - & + & - & 0 & 0 & - & 0\\
- & + & - & + & 0 & - & 0 & 0\\
- & - & + & + & - & 0 & 0 & 0
\end{pmatrix}
$$

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