Gödel's Theorem

Peter J. Cameron

Queen Mary, University of London, Mile End Road, London E1 4NS, U.K.

In response to problems in the foundations of mathematics such as *Russell's Paradox* ('Consider the set of all sets which are not members of themselves. Is it a member of itself?'), David Hilbert proposed that the consistency of a part of mathematics (such as the natural numbers) was to be established by finitary methods which could not lead to contradiction. Then this part can be used as a secure foundation for all of mathematics.

Such a branch of mathematics can be described in terms of first-order logic. We begin with symbols, logical (connectives such as 'not' and 'implies', quantifiers such as 'for all', the equality symbol, symbols for variables, and punctuation) and non-logical (symbols for constants, relations and functions suitable for the branch of mathematics under consideration.) Formulae are finite strings of symbols built according to certain rules (so that they can be mechanically recognised). We take a recognisable subset of the formulae as *axioms*, and also *rules of inference* allowing some formulae to be inferred from others. A *theorem* is a formula which is at the end of a chain (or tree) of inference starting with axioms.

Axioms for the natural numbers were given by Peano. The non-logical symbols are zero, the 'successor function' s, addition and multiplication (the last two can be defined in terms of the others by inductive axioms). The crucial axiom is the Principle of Induction, asserting that if a formula $P(n)$ is such that $P(0)$ is true and $P(n)$ implies $P(s(n))$ for all n, then $P(n)$ is true for all n. Specifically, Hilbert asked for a proof of the consistency of this theory, that is, a proof that no contradiction can be deduced from the axioms by the rules of first-order logic.

Hilbert's program was undone by two remarkable *Incompleteness Theorems* proved by Kurt Gödel:

Theorem 1. *1. There are (first-order) statements about the natural numbers which can neither be proved nor disproved from Peano's axioms (assuming that the axioms are consistent).*

2. It is impossible to prove from Peano's axioms that they are consistent.

Gödel's proof is long, but is based on two simple ideas. The first is *Gödel numbering*, where each formula or sequence of formulae is encoded by a natural number in a mechanical way. It can be shown that there is a two-variable formula $\omega(x, y)$ such that $\omega(m, n)$ holds if and only if m is the Gödel number of a formula ϕ and n the Gödel number of a proof of ϕ. Now the formula $(\forall y)(\neg\omega(x, y))$ has a Gödel number p: let ζ be the result of substituting p for x in this formula. This brings us to the second idea in the proof, self-reference: ζ asserts its own unprovability! Hence ζ is indeed unprovable, and so it is true; being true, it is not disprovable (unless the axioms are inconsistent).

It is more elementary to see that Peano's axioms are not *categorical*: even if they are consistent, there are models for the axioms which are not isomorphic to the natural numbers. Such *non-standard models* contain infinitely large numbers (bigger than all natural numbers).

The proof is not specific to the Peano axioms, but applies to any system of axioms powerful enough to describe the natural numbers. (By contrast, it is possible to find *complete* axiom systems (such that every true statement is provable) for the theory of the natural numbers with zero, successor and addition. So multiplication is essential to the argument.

Completeness cannot be restored simply by adding a true but unprovable statement as a new axiom. For the resulting system is still strong enough for Gödel's Theorem to apply to it.

Assuming that the natural numbers exist, it seems that we could obtain a complete axiomatisation by simply taking all true statements as axioms. However, one requirement of a first-order theory is that the axioms should be recognisable by some mechanical method. As Turing subsequently showed, the true statements about the natural number cannot be mechanically recognised (their Gödel numbers do not form a *recursive set*).

Gödel's true but unprovable statement is important for foundations but has no particular mathematical significance of its own. Later, Paris and Harrington gave the first example of a mathematically significant statement which is unprovable from Peano's axioms. Their statement is a variant on Ramsey's Theorem. Subsequently, many other 'natural incompletenesses' have been found.

Of course, the consistency of Peano's axioms can be proved in a stronger system. Trivially, we could just add it as an axiom: $\neg(\exists n)\omega(k, n)$ will do, where k is the Gödel number of the formula $0 = 1$. Less trivially, since a model of the natural numbers can be constructed within set theory, the consistency of Peano arithmetic can be proved from the Zermelo–Fraenkel axioms ZFC for set theory. Of course, ZFC cannot prove its own consistency, but this can be deduced from a yet stronger system (for example, adding an axiom asserting the existence of a suitably 'large' cardinal number such as an *inaccessible cardinal*).

Gödel's theorem has been a battleground for philosophers arguing about whether the human brain is a deterministic machine (in which case, presumably, we would not be able to prove any formally unprovable statement). Fortunately, space does not allow me to give more details!