Filters, topologies and groups from the random graph

Peter J. Cameron *, Sam Tarzi

School of Mathematical Sciences, Queen Mary, University of London, London E1 4NS, U.K.

Abstract

We investigate the filter generated by vertex neighbourhoods in the countable random graph R, and two related topologies, with emphasis on their automorphism groups. These include a number of highly transitive groups containing $\mathrm{Aut}(R)$.

Key words: Filter, topology, random graph, automorphism group

1 Introduction

The role of relational structures (graphs, directed graphs, partial orders, etc.) in investigating permutation groups is well-known. Of course, the automorphism group of a non-trivial relational structure cannot be highly transitive; if a group is highly transitive, we would expect to have to use "infinitary" structures such as filters and topologies in its study. In fact, the gap is not so large. For example, Macpherson and Praeger [5] showed that a permutation group of countable degree which is not highly transitive is contained in a maximal subgroup of the symmetric group. As a reviewer of the paper said, "Somewhat surprisingly, the proof is not entirely combinatorial, but also involves a little model theory, in particular, an appeal to the Cherlin–Mills–Zil'ber theorem on $\aleph_0$-categorical strictly minimal sets." The other ingredients are filters and topologies, and indeed their main task is to show that such a group preserves a non-trivial filter; they deduce this from the fact that it preserves a non-trivial topology. (A more elementary proof of part of this theorem, avoiding the model theory, was given in [3].)

* Corresponding author
Email address: p.j.cameron@qmul.ac.uk (Peter J. Cameron).

The purpose of this paper is to carry further the investigation of topologies and filters derived from relational structures, and their automorphism groups. As a case study, we take the celebrated "random graph": the next section of the paper is a brief introduction to this object. The vertex neighbourhoods in the random graph generate a non-trivial filter (indeed, it is the unique "minimal" countable graph with this property). We investigate this filter, its (highly transitive) automorphism group, and several related permutation groups. Then we turn to a topology defined from the random graph in a rather similar way. If we treat the graph and its complement symmetrically, we obtain a stronger topology homeomorphic to the rationals.

2 The random graph and its relations

In 1963 Erdős and Rényi [4] proved the remarkable result that there exists a countable graph R with the following property: if a random countable graph is chosen by selecting edges independently with probability $1/2$, then the result is isomorphic to R with probability 1. This graph and its automorphism group have received a lot of attention; some of the known results are summarised in [1]. In this paper we consider the neighbourhood filter of R and two related topologies, and their automorphism groups.

In the remainder of this section, we sketch some properties of R and of a couple of related objects: the random 3-edge-coloured complete graph, and the random bipartite graph.

The graph R has the following properties (see [1]):

- Given finite disjoint sets U and V of vertices, there exists a vertex z joined to every vertex in U and to no vertex in V. (This property characterises R up to isomorphism as a countable graph.)
- R is *universal*: every finite (or countable) graph is embeddable as an induced subgraph.
- R is *homogeneous*: every isomorphism between finite induced subgraphs of R can be extended to an automorphism of R. (This property and the preceding one also characterise R up to isomorphism as a countable graph.)
- It follows from the first property that any finite set of vertices of R has a common neighbour. This property characterises the class of countable graphs containing R as a spanning subgraph. We use this property of R, and have provided a proof in an appendix to the paper.

We can think of R as the graph formed by the red edges if the edges of the countable complete graph are randomly coloured red or blue. More generally, if the edges are randomly coloured with any finite set of colours, the resulting object (with probability 1) is uniquely determined up to isomorphism by the number k of colours used: this is called the *random k-edge-coloured complete graph.* The property which characterises this object is: given finite disjoint sets $U_1, \ldots, U_k$ of vertices, there is a vertex z such that edges from z to U_i have the ith colour for $i = 1, \ldots, k$.

Suppose that we take a countable set X of vertices partitioned into two countable subsets Y and Z. Form a random bipartite graph by choosing edges between Y and Z independently with probability 1/2. Again there is a graph B which occurs with probability 1: the *generic bipartite graph.* It has properties similar to those of R. In particular, it is not homogeneous as a graph, but if we regard it as a graph with bipartition (i.e. there are two relations, one the equivalence relation defining the two bipartite blocks and the other the adjacency relation) then it is homogeneous. Thus, the setwise stabiliser of Y in the automorphism group of B acts highly transitively (i.e. k-transitively for all k) on Y (and on Z). The graph B is characterised by a similar property: If U and V are finite disjoint subsets of a bipartite block (either Y or Z), then there is a vertex z in the other bipartite block joined to all vertices in U and to none in V.

Thomas [8] determined all the reducts of R (the closed subgroups of $\mathrm{Sym}(R)$ which contain $\mathrm{Aut}(R)$). These are as follows:

- $\mathrm{Aut}(R)$;
- $D(R)$, the group of automorphisms and anti-automorphisms of R (where an *anti-automorphism* of a graph Γ is an isomorphism from Γ to the complementary graph);
- $S(R)$, the group of switching automorphisms of R (see below);
- $B(R)$, the group of switching automorphisms and anti-automorphisms of R;
- $\mathrm{Sym}(R)$.

The operation of *switching* a graph Γ with respect to a subset Y of its vertex set $V(\Gamma)$ consists of interchanging adjacency and non-adjacency between Y and its complement $Z = V(\Gamma) \setminus Y$, while preserving adjacency and non-adjacency within Y and within Z. A *switching automorphism* of Γ is an isomorphism from Γ to a graph obtained from Γ by switching, while a *switching anti-automorphism* of Γ is an isomorphism to a graph obtained from the complement of Γ by switching. Note that the groups $D(R)$ and $S(R)$ are 2-transitive, while $B(R)$ is 3-transitive. We refer to [8] for details. On the other hand, the groups considered in this paper are not reducts: they are highly transitive, so their closure is the symmetric group.

3 Neighbourhood filters

A *filter* on a set is a family $\mathcal{F}$ of subsets of V satisfying

- $X, Y \in \mathcal{F}$ implies $X \cap Y \in F$;
- $X \in F$, $Y \supseteq X$ implies $Y \in \mathcal{F}$.

A filter $\mathcal{F}$ on a set V is *trivial* if it consists of all subsets of V; it is *principal* if it consists of all sets containing a fixed subset A of $\mathcal{F}$; and it is an *ultrafilter* if, for any $X \subseteq V$, just one of X and $V \setminus X$ belongs to $\mathcal{F}$. Ultrafilters are just maximal non-trivial filters; the axiom of choice implies that every non-trivial filter is contained in an ultrafilter.

Given a family $\mathcal{A}$ of subsets of V, the *filter generated by* $\mathcal{A}$ is the set

$$\mathcal{F} = \{X \subseteq V : (\exists A_1, \ldots, A_n \in \mathcal{A})(A_1 \cap \cdots \cap A_n) \subseteq X\}.$$

Two families $\mathcal{A}_1$ and $\mathcal{A}_2$ generate the same filter if and only if each member in $\mathcal{A}_2$ lies in the filter generated by $\mathcal{A}_1$ (that is, contains a finite intersection of sets of $\mathcal{A}_1$) and *vice versa*.

Let Γ be a graph on a countable vertex set V. We define the *neighbourhood filter* $\mathcal{F}_\Gamma$ *of* Γ *of* Γ to be the filter generated by the sets $\{\Gamma(v) : v \in V\}$, where $\Gamma(v)$ denotes the neighbourhood of v in Γ, the set of vertices adjacent to v.

Proposition 1 *Suppose that Γ has the property that each vertex has a non-neighbour. Then the filter generated by the closed neighbourhoods $\overline{\Gamma}(v) = \Gamma(v) \cup \{v\}$ is equal to $\mathcal{F}_\Gamma$.*

PROOF. We have $\Gamma(v) \subseteq \overline{\Gamma}(v)$, and, if w is not adjacent to v, then $\overline{\Gamma}(v) \cap \overline{\Gamma}(w) \subseteq \Gamma(v)$.

The condition on Γ is necessary. If Γ is the complete graph, the closed neighbourhoods generate the filter $\{V\}$, while the open neighbourhoods generate the filter of cofinite subsets of V.

Let R denote the countable random graph.

Proposition 2 *The following three conditions on a graph Γ are equivalent:*

(a) *$\mathcal{F}_\Gamma$ is nontrivial;*
(b) *Γ contains R as a spanning subgraph;*

(c) $\mathcal{F}_\Gamma \subseteq \mathcal{F}_R$.

PROOF. A filter is trivial if and only if it contains the empty set. So $\mathcal{F}_\Gamma$ is non-trivial if and only if any finitely many neighbourhoods have non-empty intersection. This is equivalent to the statement that R is a spanning subgraph of Γ, as noted in the last section. So (a) and (b) are equivalent.

If Γ contains R as a spanning subgraph, then $R(v) \subseteq \Gamma(v)$ for all v. So (b) implies (c). Conversely, $\mathcal{F}_R$ is non-trivial (by our proof that (b) implies (a)), so (c) implies (a).

Remark This result shows that $\mathcal{F}_R$ is the unique maximal neighbourhood filter. But this uniqueness is only up to isomorphism. So part (c) really means that $\mathcal{F}_\Gamma$ is contained in a filter isomorphic to $\mathcal{F}_R$.

For example, it is possible to find two filters isomorphic to $\mathcal{F}_R$, one contained in the other. For let T be the random 3-colouring of the edges of the complete graph, with colours red, green and blue. Let R_1 be the graph consisting of red edges, and R_2 the graph consisting of red and green edges, in T. Then both R_1 and R_2 are isomorphic to R. Since $R_1(v) \subseteq R_2(v)$, we have $\mathcal{F}_{R_2} \subseteq \mathcal{F}_{R_1}$. We show that the inequality is strict.

The set $R_1(v)$ belongs to $\mathcal{F}_{R_1}$. Suppose that it belongs to $\mathcal{F}_{R_2}$. Then there are vertices $w_1, \ldots, w_n$ such that

$$R_2(w_1) \cap \ldots \cap R_2(w_n) \subseteq R_1(v).$$

But, since the green graph is isomorphic to R, there is a vertex x joined to all of $v, w_1, \ldots, w_n$ by green edges; then x belongs to the left-hand expression of the displayed inclusion but not to $R_1(v)$, a contradiction.

Similarly it can be shown that there are countable chains of filters isomorphic to $\mathcal{F}_R$.

4 Groups

Clearly $\mathrm{Aut}(R)$ is a subgroup of $\mathrm{Aut}(\mathcal{F}_R)$. We will see in this section that $\mathrm{Aut}(\mathcal{F}_R)$ is much larger than $\mathrm{Aut}(R)$.

First we define a few groups. We say that a permutation g changes the adjacency of v and w if $(v \sim w) \Leftrightarrow (v^g \not\sim w^g)$. We say that g changes finitely many adjacencies at v if there are only finitely many points w for which g changes the adjacency of v and w. Let $C(g)$ be the set of pairs $\{v, w\}$ whose adjacency is changed by g. Then $C(g^{-1}) = C(g)^{g^{-1}}$ and $C(gh) \subseteq C(g) \cup C(h)^{g^{-1}}$.

- $\mathrm{Aut}_1(R)$ is the group of permutations which change only finitely many adjacencies (these are called *almost-automorphisms*, and Truss [9] denotes the group by $\mathrm{AAut}(R)$);
- $\mathrm{Aut}_2(R)$ is the group of permutations which change only finitely many adjacencies at any vertex;
- $\mathrm{Aut}_3(R)$ is the group of permutations which change infinitely many adjacencies at only finitely many vertices;
- $\mathrm{FSym}(V)$ and $\mathrm{Sym}(V)$ are the finitary symmetric group and the full symmetric group on the set V.

A little thought shows that all these sets of permutations really are groups, as claimed. (For $\mathrm{Aut}_i(R)$, use the above facts about $C(g)$.)

Proposition 3 *(a)* $\mathrm{Aut}_2(R) \leq \mathrm{Aut}(\mathcal{F}_R)$.
(b) Neither of $\mathrm{Aut}_3(R)$ *and* $\mathrm{Aut}(\mathcal{F}_R)$ *contains the other.*
(c) $\mathrm{FSym}(V) \leq \mathrm{Aut}_3(R) \cap \mathrm{Aut}(\mathcal{F}_R)$, *but* $\mathrm{FSym}(V) \cap \mathrm{Aut}_2(R) = \{1\}$.

PROOF. (a) Let $g \in \mathrm{Aut}_2(R)$. It suffices to show that, for any vertex v, we have $R(v)^g \in \mathcal{F}_R$. Now by assumption, $R(v)^g$ differs only finitely from $R(v^g)$; let $R(v)^g \setminus R(v^g) = \{x_1, \ldots, x_n\}$. Then we have

$$R(v^g) \cap R(x_1) \cap \cdots \cap R(x_n) \subseteq R(v)^g,$$

and we are done.

(b) Choose a vertex v, and consider the graph R' obtained by changing all adjacencies at v. Then $R' \cong R$. Choose an isomorphism g from R to R'; since R' is vertex-transitive, we can assume that g fixes v. So g maps $R(v)$ to $R_1(v) = V \setminus (\{v\} \cup R(v))$. Clearly $g \in \mathrm{Aut}_3(R)$, since it changes only one adjacency at any point different from v. But if $g \in \mathrm{Aut}(\mathcal{F}_R)$, then we would have $R_1(v) \in \mathcal{F}_R$, a contradiction since $R(v) \cap R_1(v) = \emptyset$.

In the reverse direction, let R'' be the graph obtained by changing all adjacencies between non-neighbours of v. Again $R'' \cong R$, and we can pick an isomorphism from R to R'' which fixes v. Now g changes infinitely many adjacencies at all non-neighbours of v (and none at v or its neighbours). Also, if w is a non-neighbour of v, then $R(v) \cap R(w)^g = R(v) \cap R(w^g)$, so $g \in \mathrm{Aut}(\mathcal{F}_R)$.

(c) Any non-identity finitary permutation belongs to $\mathrm{Aut}_3(R) \setminus \mathrm{Aut}_2(R)$. For if g moves v, then g changes infinitely many adjacencies at v (namely, all v

and w, where w is adjacent to v but not v^g and is not in the support of g). On the other hand, if g fixes v, then v changes the adjacency of v and w only if g moves w, and there are only finitely many such w.

Finally, if $g \in \mathrm{FSym}(V)$, then $R(v)^g$ differs only finitely from $R(v)$, for any vertex $v \in V$; so $g \in \mathrm{Aut}(\mathcal{F}_R)$.

The inclusion in (c) is proper: $\mathrm{Aut}_2(R)$ is contained in the right-hand side but intersects $\mathrm{FSym}(V)$ in $\{1\}$.

The graph R' in the proof of (b) is obtained from R by switching with respect to the set $\{v\}$; so the permutation g belongs to the group $S(R)$ of switching automorphisms. Thus $S(R) \not\leq \mathrm{Aut}(\mathcal{F}_R)$. In fact, more is true:

Proposition 4 $\mathrm{Aut}(\mathcal{F}_R) \cap D(R) = \mathrm{Aut}(\mathcal{F}_R) \cap S(R) = \mathrm{Aut}(R)$.

PROOF. Any anti-automorphism g of R maps $R(v)$ to a set disjoint from $R(v^g)$; so no anti-automorphism can belong to $\mathrm{Aut}(\mathcal{F}_R)$.

Suppose that $g \in \mathrm{Aut}(\mathcal{F}_R)$ is an isomorphism from R to $\sigma_X(R)$, where σ_X denotes switching with respect to X. We may suppose that σ_X is not the identity, that is, $X \neq \emptyset$ and $Y = V \setminus X \neq \emptyset$. Choose x and y so that $x^g \in X$ and $y^g \in Y$. Then $R(x)^g \bigtriangleup Y = R(x^g)$ and $R(y)^g \bigtriangleup X = R(y^g)$. Hence $R(x^g) \cap R(x)^g \subseteq X$ and $R(y^g) \cap R(y^g) \subseteq Y$. Hence

$$R(x^g) \cap R(x)^g \cap R(y^g) \cap R(y)^g = \emptyset,$$

a contradiction.

5 Topology

If $\mathcal{F}$ is a filter, then $\mathcal{F} \cup \{\emptyset\}$ is a topology with the same automorphism group. In this section we discuss two further topologies on $V(R)$ defined from the graph R.

In the first topology $\mathcal{T}$, a sub-basis for the open sets consists of the open neighbourhoods of vertices. Thus the open sets are all unions of sets which are finite intersections of open neighbourhoods.

The topology $\mathcal{T}$ is not Hausdorff: in fact, any two open sets have non-empty intersection. For it suffices to show this for basic open sets; and the intersection of two finite intersections of neighbourhoods is itself a finite intersection of neighbourhoods, and so is non-empty.

However, this topology does satisfy the T1 separation condition: for, given distinct points x and y, there is a vertex v joined to x but not y, and so a neighbourhood containing x but not y. Hence all singletons (and so all finite sets) are closed.

We used open neighbourhoods in the construction of $\mathcal{T}$. In fact, closed neighbourhoods would have given us the same topology, as we will now see.

Let B denote the generic bipartite graph. Consider the three topologies which have the following as the points and sub-basic open sets:

$\mathcal{T}$: points are vertices of R, sub-basic open sets are open vertex neighbourhoods.
$\mathcal{T}^*$: points are vertices of R, sub-basic open sets are closed vertex neighbourhoods.
$\mathcal{T}^\dagger$: points are one bipartite block in B, sub-basic open sets are neighbourhoods of vertices in the other bipartite block.

Proposition 5 *(a) The three topologies defined above are all homeomorphic.*
(b) The homeomorphism groups of these topologies are highly transitive.

PROOF. From R, we construct two bipartite graphs B_1 and B_2 as follows. The vertex set of each graph is $V(R) \times \{0,1\}$; vertices $(v,0)$ and $(w,1)$ are adjacent if and only if

- $v \sim w$ in R (for B_1);
- $v = w$ or $v \sim w$ in R (for B_2).

The characteristic property of R shows that both bipartite graphs satisfy the characteristic property of the generic bipartite graph B; so $B_1 \cong B_2 \cong B$. It follows immediately that the three topologies are homeomorphic.

Moreover, the stabiliser in $\mathrm{Aut}(B)$ of a bipartite block acts on it as a group of homeomorphisms of the topology $\mathcal{T}^\dagger$, and this group is highly transitive. (This follows from the homogeneity of B as a graph with bipartition: any two vertices in the same bipartite block have distance 2, so any bijection between finite subsets of a bipartite block extends to an automorphism of B).

Remark The topologies $\mathcal{T}$ and $\mathcal{T}^*$, though homeomorphic, are not identical. Indeed, the identity map is a continuous bijection from $\mathcal{T}^*$ to $\mathcal{T}$ but not a homeomorphism.

To see this, note first that, since the topology $\mathcal{T}^*$ is T1, every singleton is closed, and so $R(v) = (R(v) \cup \{v\}) \setminus \{v\}$ is open in $\mathcal{T}^*$. It follows that any open set in $\mathcal{T}$ is also open in $\mathcal{T}^*$.

In the other direction, suppose that $R(v) \cup \{v\}$ is open in $\mathcal{T}$. Then it is a union of basic open sets. We can take one of these sets to be $R(v)$; let the other be $\bigcap_{x \in X} R(x)$ for some finite set X. Then v is joined to all vertices in X, but the remaining common neighbours of these vertices are all in $R(v)$. So no point is joined to all vertices in X but not to v, a contradiction.

The second topology $\mathcal{U}$ is obtained by symmetrising this one with respect to the graph R and its complement R^c; in other words, we also take closed neighbourhoods in R^c to be open sets. So a basis for the open sets consists of all sets of the form

$$Z(U, V) = \{z \in V(R) : (\forall u \in U)(z \sim u) \wedge (\forall v \in V)(z \not\sim v)\}$$

for finite disjoint sets U and V. Again it holds that all the non-empty open sets are infinite. This time the topology is totally disconnected. For given $u \neq v$, there is a point $z \in Z(\{u\}, \{v\})$; then the open neighbourhood of z is open and closed in the topology and contains u but not v.

By Sierpiński's Theorem stated below ([7], see also [6]), this topology is homeomorphic to $\mathbb{Q}$. So R as a countable topological space is homeomorphic to $\mathbb{Q}$.

Theorem 6 *Let $\mathcal{U}$ be a countable, second countable, totally disconnected topological space with no isolated points. Then $\mathcal{U}$ is homeomorphic to the usual topology on $\mathbb{Q}$.*

We end this section with several questions about the topology $\mathcal{T}$.

- Since $\mathcal{T}$ is a coarsening of $\mathcal{U}$, there must be an identification of it with $\mathbb{Q}$ such that the open sets in $\mathcal{T}$ are open in $\mathbb{Q}$. Can such an identification be found explicitly?
- Is there a characterisation of $\mathcal{T}$, along the lines of Sierpiński's Theorem?
- The homeomorphism group $\mathrm{Aut}(\mathcal{T}^\dagger)$ contains the group $\mathrm{Aut}'(B)$ induced on a bipartite block of B by its setwise stabiliser in the automorphism group of B. This group is highly transitive. Is it equal to $\mathrm{Aut}(T^\dagger)$?

We cannot answer these questions, but we present here a programme which might lead to an affirmative answer for the third question (which would have implications for the second as well).

Call an open set U *full* if, for all $x \notin U$, the set $U \cup \{x\}$ is not open. By the argument used previously to show that the topologies $\mathcal{T}$ and $\mathcal{T}^*$ are different, we see that any positive Boolean combination of neighbourhoods in $\mathcal{T}^\dagger$ (that is, any finite union of basic open sets) is full. *Is it true that these are the only full open sets in $\mathcal{T}^\dagger$?*

If so, then we can recognise the basic open sets as the full open sets which are not proper finite unions of full open sets; and then the neighbourhoods are the basic open sets which are maximal under inclusion. So we can recover the graph B from the topology, and every homeomorphism is a graph automorphism.

Finally, we can ask: *How are the homeomorphism groups of $\mathcal{T}$ and $\mathcal{T}^*$ related to the groups of Section 4?* Since $\mathcal{F}_R$ consists of all sets containing a non-empty $\mathcal{T}$-open set, we see that $\mathrm{Aut}(\mathcal{T}) \leq \mathrm{Aut}(\mathcal{F}_R)$; do any further relations hold?

6 Other graphs

Given a graph Γ, we define the *k-neighbourhood* of a vertex v in Γ to be the set of points distant at most k from v. Which graphs have the property that their k-neighbourhoods generate a non-trivial filter, for some fixed k? It is easy to see that such a graph has diameter at most $2k$ (else two k-neighbourhoods are disjoint). Moreover, we can assume that the diameter is at least $k+1$ (else every k-neighbourhood is the whole vertex set).

We make the following observation. For every positive integer d, there is a countable homogeneous universal *integral metric space* (one with integer distances) of diameter d, unique up to isometry: see [2]. The metric is the path metric in a graph M_d of diameter d. Thus, M_2 is the random graph R.

It is easy to show that the filter generated by the k-neighbourhoods in M_{2k} is isomorphic to $\mathcal{F}_R$.

Is the following true? *Let Γ be a countable graph whose k-neighbourhoods generate a non-trivial filter. Then M_{2k} is a spanning subgraph of Γ.*

What can be said about other distance classes?

Appendix

Here is the proof of the fact that a countable graph Γ contains R as a spanning subgraph if and only if any finite set of vertices has a common neighbour in Γ.

Let $V(\Gamma) = \{v_0, v_1, \ldots\}$ and $V(R) = \{w_0, w_1, \ldots\}$. We construct a bijection ϕ from $V(R)$ and $V(\Gamma)$ by back-and-forth.

At even-numbered stages, choose the first unused vertex w of R. Let U and V be the sets of neighbours and non-neighbours of w among vertices on which ϕ has been defined. Choose $v \in V(\Gamma)$ joined to all vertices in $\phi(U)$, and extend ϕ to map w to v. In this extension, edges are mapped to edges.

At odd-numbered stages, choose the first unused vertex v of Γ. Let U and V be its neighbours and non-neighbours among vertices in the image of ϕ. Choose a vertex w of R joined to no vertex in $\phi^{-1}(V)$, and extend ϕ to map w to v. Again, edges map to edges since the inverse image of a non-edge is a non-edge. After countably many steps we have the required bijection which takes edges of R to edges of Γ.

References

[1] P. J. Cameron, The random graph, pp. 331–351 in *The Mathematics of Paul Erdős*, (J. Nešetřil and R. L. Graham, eds.), Springer, Berlin, 1996.

[2] P. J. Cameron, A census of infinite distance-transitive graphs, *Discrete Math.* **192** (1998), 11–26.

[3] P. J. Cameron, Topology in permutation groups, in *Groups: Topological, Combinatorial and Arithmetic Aspects* (ed. T. W. Müller), London Math. Soc. Lecture Notes **311**, Cambridge University Press, Cambridge, 2004, pp. 93–105.

[4] P. Erdős and A. Rényi, Asymmetric graphs, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 295–315.

[5] H. D. Macpherson and C. E. Praeger, Maximal subgroups of infinite symmetric groups. *J. London Math. Soc.* (2) **42** (1990), 85–92.

[6] P. M. Neumann, Automorphisms of the rational world, *J. London Math. Soc.* (2) **32** (1985), 439–448.

[7] W. Sierpiński, Une propriété topologique des ensembles dénombrables denses en soi, *Fund. Math.* **1** (1920), 11–16.

[8] S. R. Thomas, Reducts of the random graph, *J. Symbolic Logic* **56** (1991), 176–181.

[9] J. K. Truss, The group of almost automorphisms of the countable universal graph, *Math. Proc. Cambridge Philos. Soc.* **105** (1989), 223–236.