Limits of cubes

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Abstract

The celebrated Urysohn space is the completion of a countable universal homogeneous metric space which can itself be built as a direct limit of finite metric spaces. It is our purpose in this paper to give another example of a space constructed in this way, where the finite spaces are scaled cubes. The resulting countable space provides a context for a direct limit of finite symmetric groups with strictly diagonal embeddings, acting naturally on a module which additively is the "Nim field" (the quadratic closure of the field of order 2). Its completion is familiar in another guise: it is the set of Lebesgue-measurable subsets of the unit interval modulo null sets. We describe the isometry groups of these spaces and some interesting subgroups, and give some generalisations and speculations.

Key words: metric space; isometry group; completion; cube

1 Introduction

In this paper we discuss several constructions of metric spaces which are limits of sequences of isometric embeddings of scaled finite cubes. We mainly concentrate on the case where each embedding doubles the dimension; but we begin with a simpler example where we just add one to the dimension, and in the concluding section we consider briefly an embedding in which the n-dimensional cube is embedded in the cube of dimension $n!$.

1.1 Finite cubes

The n-dimensional cube, or *Hamming space* over the alphabet $\{0, 1\}$, is the set of all binary n-tuples, endowed with *Hamming distance*, where the distance between two n-tuples v and w is equal to the number of coordinates where they differ. We can of course identify it with the power set of a standard n-element set $\{0, 1, \ldots, n-1\}$; the distance between two sets A and B is the cardinality of their symmetric difference $A \triangle B$. We denote it by $H(n, 2)$.

The isometry group $\mathrm{Iso}(H(n,2))$ is the wreath product $\mathrm{Sym}(2) \wr \mathrm{Sym}(n)$, which is a semidirect product $T_n \rtimes S_n$ of an elementary abelian group T_n of order 2^n by the symmetric group S_n.

1.2 A limit of finite cubes

The obvious way to create an "infinite cube" is simply to let the length n of the tuples become infinite. Of course, if we do that, then the distances between tuples can also be infinite. In order to obtain a metric space, we can take the set of tuples containing only finitely many ones (or equivalently, the finite subsets of $\mathbb{N}$). This is a discrete metric space of infinite diameter, whose isometry group is a semidirect product $T \rtimes S$, where T is a countable elementary abelian 2-group and $S = \mathrm{Sym}(\mathbb{N})$.

Another way of looking at this is to observe that there are isometric embeddings

$$H(1,2) \to H(2,2) \to H(3,2) \to \cdots$$

where $H(n,2)$ is embedded in $H(n+1,2)$ as the set of points with last coordinate zero. Taking the union, we obtain the cube as above. The embeddings of metric spaces induce embeddings

$$G_1 \to G_2 \to G_3 \to \cdots$$

where $G_n = T_n \rtimes S_n$ as above. The union of these groups is a proper subgroup of the full isometry group, of the form $T \rtimes S_0$, where S_0 is the group of all finitary permutations of $\mathbb{N}$, the union of the chain $S_1 \to S_2 \to S_3 \to \cdots$ of finite symmetric groups. (Note that we do get the full translation group in the union.) The closure of G_0 (in the topology of pointwise convergence) is the full isometry group G.

2 Another limit of finite cubes

The "infinite cube" constructed in the preceding section is discrete. In this section we construct a different limit of finite cubes, whose completion is more interesting.

2.1 Construction of $\mathcal{H}_\omega$ and $\mathcal{H}$

Given positive real numbers $c_0, \ldots, c_{n-1}$, we define the *scaled hypercube* $(c_0, \ldots, c_{n-1})H(n, 2)$ to have the same set of points, but with distance given by

$$d(x, y) = \sum_{i:x_i \neq y_i} c_i.$$

If $c_0 = \cdots = c_{n-1} = c$, we write simply $cH(n, 2)$. Furthermore, we denote $\frac{1}{n}H(n, 2)$ by $\mathcal{H}_n$. This metric space has diameter 1.

There is an isometric embedding $\theta : \mathcal{H}_n \to \mathcal{H}_{2n}$ given by

$$(\theta(x))_{2i} = (\theta(x))_{2i+1} = x_i, \qquad i = 0, \ldots, n-1.$$

This embedding has the further property that every isometry of $\mathcal{H}_n$ is induced by an isometry of $\mathcal{H}_{2n}$.

We can iterate this embedding to get a chain

$$\mathcal{H}_1 \to \mathcal{H}_2 \to \mathcal{H}_4 \to \mathcal{H}_8 \to \cdots$$

Let $\mathcal{H}_\omega$ denote the union of the chain, and $\mathcal{H}$ its completion. These metric spaces are our candidates for an infinite-dimensional cube. It follows from our remarks that every isometry of any term in the chain is induced by an isometry of $\mathcal{H}_\omega$ (and hence of $\mathcal{H}$).

2.2 $\mathcal{H}_\omega$ as a Cayley metric space: the countable Nim group

The metric space $\mathcal{H}_\omega$ can be described as a *Cayley metric space*, as follows.

Define the operation $\oplus$ of *Nim addition* (or *bitwise addition*) on the set $\mathbb{N}$ of natural numbers (including 0) as follows: to calculate $x \oplus y$, express x and y in base 2, and add modulo 2 (without carrying). Then $(\mathbb{N}, \oplus)$ is an elementary abelian 2-group. In order to define a metric on $\mathbb{N}$ invariant under this group, it suffices to find a function f from $\mathbb{N}$ to the positive real numbers satisfying

$$|f(x) - f(y)| \leq f(x \oplus y) \leq f(x) + f(y) \text{ for all } x, y \in \mathbb{N},$$

and then put $d(x, y) = f(x \oplus y)$ for all $x, y \in \mathbb{N}$.

We define a function f recursively as follows:

$$f(0) = 0, \qquad f(1) = 1,$$
$$f(2^{2^n}x + y) = \tfrac{1}{2}(f(y) + f(x \oplus y)) \text{ for } 0 \leq x, y \leq 2^{2^n} - 1.$$

(Note that any number $z > 1$ satisfies $2^{2^n} \leq z < 2^{2^{n+1}}$ for some unique n, and such a z can be uniquely written in the form $2^{2^n}x + y$ for $0 \leq x, y < 2^{2^n}$.) The recursive statement extends the range of definition of f from the set $\{0, \ldots, 2^{2^n} - 1\}$ to $\{0, \ldots, 2^{2^{n+1}} - 1\}$. The function is well-defined since the right-hand side of the recursion gives $f(y)$ when $x = 0$.

Proposition 1 *If the function f is defined by the above recursion, then the induced Cayley metric on $\mathbb{N}$ is isometric to $\mathcal{H}_\omega$.*

PROOF. Clearly the induced metric on $\{0, 1\}$ is isometric to $\mathcal{H}_1$.

Suppose that the induced metric on $\{0, \ldots, 2^{2^n} - 1\}$ is isometric to $\mathcal{H}_{2^n}$, via an isometry ϕ_n. Then the composition of ϕ_n with the coordinate-doubling map θ is an isometry to a subspace of $\mathcal{H}_{2^{n+1}}$. We have to show that this map extends to an isometry from $\{0, \ldots, 2^{2^{n+1}} - 1\}$ to $\mathcal{H}_{2^{n+1}}$. Because the initial set is a subgroup of the abelian group on $\mathbb{N}$, it is enough to consider the function f.

We have embedded $\mathcal{H}_{2^n}$ into $\mathcal{H}_{2^{n+1}}$ as the set of points y satisfying $y_{2i} = y_{2i+1}$ for $i = 0, \ldots, 2^n - 1$. Any point can be expressed uniquely as $x + y$ (with coordinatewise addition mod 2), where y is as above and x is supported on the even coordinates: that is, $x_{2i+1} = 0$ for $i = 0, \ldots, 2^n - 1$. We have to show that

$$\frac{1}{2^{n+1}} s(x + y) = \tfrac{1}{2}(f^*(y) + f^*(x \oplus y)),$$

where $f^*(x) = f(\phi_n^{-1}(x))$, and s is the support size. By the induction hypothesis, $f^*(y)$ is equal to the size of the support of y, divided by 2^n.

Let A, B and C be pairwise disjoint subsets of $\{0, \ldots, 2^n - 1\}$ such that the supports of x and y are $\{2u : u \in A \cup B\}$ and $\{2u, 2u + 1 : u \in A \cup C\}$ respectively, and let $f^*(x)$ denote $f(\phi_n^{-1}(x)) = \frac{1}{2^n} s(x)$. We have

$$\begin{aligned} f^*(x) &= \frac{1}{2^n}(|A| + |B|), \\ f^*(y) &= \frac{1}{2^n}(|A| + |C|), \\ f^*(x \oplus y) &= \frac{1}{2^n}(|B| + |C|). \end{aligned}$$

Thus

$$f(2^{2^n}x+y)=\frac{1}{2^{n+1}}(|A|+|B|+2|C|)=\tfrac{1}{2}(f^*(y)+f^*(x\oplus y)),$$

as required. $\square$

For example, the first 256 values of f are given in the following table, with $16x+y$ in row x and column y:

0	1	1/2	1/2	1/2	1/2	1/2	1/2	1/4	3/4	1/4	3/4	1/4	3/4	3/4	1/4
1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
1/4	3/4	1/4	3/4	1/2	1/2	1/2	1/2	1/4	3/4	1/4	3/4	1/2	1/2	1/2	1/2
1/4	3/4	3/4	1/4	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/4	3/4	3/4	1/4
1/4	3/4	1/2	1/2	1/4	3/4	1/2	1/2	1/4	3/4	1/2	1/2	1/4	3/4	1/2	1/2
1/4	3/4	1/2	1/2	3/4	1/4	1/2	1/2	1/2	1/2	1/4	3/4	1/2	1/2	3/4	1/4
1/4	3/4	1/2	1/2	1/2	1/2	1/4	3/4	1/2	1/2	1/4	3/4	1/4	3/4	1/2	1/2
1/4	3/4	1/2	1/2	1/2	1/2	3/4	1/4	1/4	3/4	1/2	1/2	1/2	1/2	3/4	1/4
1/8	7/8	3/8	5/8	3/8	5/8	5/8	3/8	1/8	7/8	3/8	5/8	3/8	5/8	5/8	3/8
3/8	5/8	5/8	3/8	5/8	3/8	3/8	5/8	5/8	3/8	3/8	5/8	3/8	5/8	5/8	3/8
1/8	7/8	3/8	5/8	5/8	3/8	3/8	5/8	3/8	5/8	1/8	7/8	3/8	5/8	5/8	3/8
3/8	5/8	5/8	3/8	3/8	5/8	5/8	3/8	3/8	5/8	5/8	3/8	3/8	5/8	5/8	3/8
1/8	7/8	5/8	3/8	3/8	5/8	3/8	5/8	3/8	5/8	3/8	5/8	1/8	7/8	5/8	3/8
3/8	5/8	3/8	5/8	5/8	3/8	5/8	3/8	3/8	5/8	3/8	5/8	5/8	3/8	5/8	3/8
3/8	5/8	3/8	5/8	3/8	5/8	3/8	5/8	3/8	5/8	3/8	5/8	3/8	5/8	3/8	5/8
1/8	7/8	5/8	3/8	5/8	3/8	5/8	3/8	3/8	5/8	3/8	5/8	3/8	5/8	7/8	1/8

In this representation, 1 is the point antipodal to 0, and more generally the points $2i$ and $2i+1$ are antipodal for all $i \in \mathbb{N}$.

Problem Is it possible to give an explicit (non-recursive) formula for $f(n)$?

The metric space $\mathcal{H}$ is also a Cayley metric space, that is, has an abelian group structure. This will follow from a more detailed look at translations in the next section.

Conway [3] has shown that the group structure defined on $\mathbb{N}$ by Nim addition is the additive group of a field, whose multiplication is given by the rules

$$\begin{aligned} 2^{2^n} \otimes y &= 2^{2^n} \cdot y \text{ for } y < 2^{2^n}, \\ 2^{2^n} \otimes 2^{2^n} &= 3 \cdot 2^{2^n - 1}. \end{aligned}$$

This field is the quadratic closure of GF(2), that is, the union of the chain

$$\mathrm{GF}(2^{2^0}) \to \mathrm{GF}(2^{2^1}) \to \mathrm{GF}(2^{2^2}) \to \mathrm{GF}(2^{2^3}) \to \cdots$$

This raises several questions about the relationship between Conway's construction and ours.

(a) Clearly the multiplication does not preserve the function f, since 1 is the unique point n satisfying $f(n) = 1$.

There is a Frobenius map $n \mapsto n^2$ on the field. Unfortunately this does not preserve the function f either, since $8^2 = 13$ but $f(8) = \frac{1}{4}$ and $f(13) = \frac{3}{4}$.

Is it true that $f(n^2) \in \{f(n), 1 - f(n)\}$ for all n?

(b) Conway [3] extended the definition of the Nim addition and multiplication to transfinite ordinals, obtaining an algebraically closed "Nim-field". Is there any connection between this field and our space $\mathcal{H}$, with its uncountable elementary abelian group of translations?

2.3 Another description: measurable sets

A more explicit description of the cubes is given by the following result.

Theorem 2 *(a) The points of $\mathcal{H}_\omega$ can be identified with the subsets of $[0, 1)$ which are unions of finitely many half-open intervals $[x, y)$ with dyadic rational endpoints, the distance between two such sets being the sum of the lengths of their symmetric difference.*

(b) The points of $\mathcal{H}$ can be identified with the Lebesgue measurable subsets of $[0, 1]$ modulo null sets, the distance between two points being the Lebesgue measure of their symmetric difference.

PROOF. (a) A point of $\mathcal{H}_{2^n}$ is a subset $A \subseteq \{0, \ldots, 2^n - 1\}$, and can be represented in the above form as

$$\bigcup_{i \in A} \left[\frac{i}{2^n}, \frac{i+1}{2^n} \right).$$

The isometric embedding of $\mathcal{H}_{2^n}$ into $\mathcal{H}_{2^{n+1}}$ corresponds to inclusion on the set of subsets of this form. Finally, given any finite union of intervals with dyadic rational endpoints, there is a maximum denominator 2^n which occurs, so the set represents a point of $\mathcal{H}_{2^n}$.

(b) In the representation of (a), it is clear that the limit of a Cauchy sequence of sets of the form given there is a measurable set (well-defined up to a null set). Conversely, let A be any measurable set. Given $\epsilon > 0$, we can find sets B and C which are unions of intervals, with $B \subseteq A \subseteq C$ and $\mu(C \setminus B) < \epsilon/2$. We can replace B and C by sets B' and C' which are finite unions of sets with dyadic rational endpoints, changing the measure of each by at most $\epsilon/4$. We obtain elements of $\mathcal{H}_\omega$ within ϵ of A. $\square$

2.4 Translations

We now turn to the isometry groups of our spaces. We define a *translation* of a metric space to be an isometry g satisfying $g^2 = 1$ such that $d(x, x^g)$ is independent of x. In general, translations have no particular properties: for example, in a discrete space with n points pairwise at distance 1, the translations are all the fixed-point-free involutions in $\mathrm{Sym}(n)$, together with the identity. However, in our cubes, they are much better behaved.

Theorem 3 *In any of the spaces $\mathcal{H}_n$, $\mathcal{H}_\omega$, or $\mathcal{H}$, the translations form an elementary abelian 2-group which is a regular normal subgroup of the full isometry group. Moreover, the translation group of $\mathcal{H}_\omega$ is the union of the translation groups of $\mathcal{H}_{2^n}$, while the translation group of $\mathcal{H}$ is the closure of that of $\mathcal{H}_\omega$ (in the topology of pointwise convergence).*

The proof depends on a lemma about the combinatorial structure of these metric spaces. Note that each such space has diameter 1, and is antipodal, with a unique point at distance 1 from any given point. For any two points x, y, the *interval* $[x, y]$ is defined as the set

$$\{z : d(x, z) + d(z, y) = d(x, y)\}.$$

Lemma 4 *Let M denote one of $\mathcal{H}_n$, $\mathcal{H}_\omega$ or $\mathcal{H}$.*

(a) Given $x, y, z \in M$, with $z \in [x, y]$, there is a unique point $w \in [x, y]$ such that $d(z, w) = d(x, y)$, $d(y, w) = d(x, z)$, and $d(x, w) = d(y, z)$. (We denote the point w by $f(x, y, z)$.)

(b) Given any three points $x, y, z \in M$, there exist unique points $z_1, z_2 \in M$ such that $z_1 \in [x, y]$, $x \in [y, z_2]$, and $z_1, z_2 \in [x, z]$ with $f(x, z, z_1) = z_2$.

PROOF. (a) In $\mathcal{H}_n$, an interval is a scaled cube, and $f(x, y, z)$ is the point antipodal to z in $[x, y]$. In $\mathcal{H}_\omega$, any three points lie in a finite cube, and the result for $\mathcal{H}_n$ applies. Finally, take $x, y, z \in \mathcal{H}$ with $z \in [x, y]$, and let (x_n), (y_n), (z_n) be Cauchy sequences in $\mathcal{H}_\omega$ converging to x, y, z respectively. It need not be the case that $z_n \in [x_n, y_n]$; but, considering a finite cube containing these points, we see that there is a unique point $z'_n \in [x_n, y_n]$ nearest to z_n, and that $d(z_n, z'_n) \to 0$. Then $z'_n \to z$ also. By the result for $\mathcal{H}_\omega$, there are points $w_n = f(x_n, y_n, z'_n)$. Then (w_n) is a Cauchy sequence, and its limit is the required point $f(x, y, z)$.

(b) First consider $\mathcal{H}_n$, represented as the power set of $\{0, \ldots, n-1\}$, we may choose $x = \emptyset$ without loss of generality. Then $z_1 = z \cap y$ and $z_2 = z \setminus y$ are the required points. The extension to $\mathcal{H}_\omega$ and $\mathcal{H}$ are almost identical to part (a). □

Now we turn to the proof of the Theorem. We observe first that in each case there is a regular elementary abelian group of translations: in the set representations, the translations are the maps $x \mapsto x \triangle a$ for fixed a. We have to show that these subgroups contain all the translations. By the regularity, it suffices to show that, given any two points x, y, there is a unique translation carrying x to y.

So let g be a translation with $x^g = y$, and let z be any further point.

- If $z \in [x, y]$, then $d(x, z^g) = d(x^g, z) = d(y, z)$; $d(y, x^g) = d(y^g, z) = d(x, z)$; and $d(z, z^g) = d(x, x^g) = d(x, y)$. So $z^g = f(x, y, z)$.
- If $x \in [y, z]$, a similar argument shows that $z^g = f(y, z, x)$.
- Suppose that neither of the above hold. Choose z_1 and z_2 as in the Lemma. Then z_1^g and z_2^g are uniquely determined, by the two cases just considered; and the facts that $z = f(z_1, z_2, x)$ and g is an isometry show that $z^g = f(z_1^g, z_2^g, y)$.

In particular, we see that the set T of all translations is a group (a regular subgroup of the full isometry group).

Finally, the definition shows that the set of translations is closed under conjugation by any isometry, and so T is a normal subgroup of the isometry group. □

2.5 The full isometry group

The theorem of the last subsection shows that the full isometry group of any of the spaces $\mathcal{H}_n$, $\mathcal{H}_\omega$ or $\mathcal{H}$ has the structure of a semidirect product

$G = T \rtimes G_0$, where T is the group of translations and G_0 is the stabiliser of a point 0 (which, without loss of generality, we can take to be the empty set in our set representation).

In the case of $\mathcal{H}_n$, as is well known, G_0 is the symmetric group $\mathrm{Sym}(n)$.

For $\mathcal{H}_\omega$, we have the following:

Proposition 5 *Any isometry of $\mathcal{H}_\omega$ fixing 0 is induced by a permutation of $[0,1]$, uniquely determined except on the dyadic rationals.*

PROOF. Let x be a real number in $[0,1)$ which is not a dyadic rational, and express x in base 2. The first n bits of x specify the interval $[k/2^n, (k+1)/2^n)$ in which x lies. The image of this interval under a fixed isometry g is a union of, say, 2^l intervals of length $1/2^{n+l}$. Now g^{-1} maps each of these intervals to a union of finitely many subintervals of $[k/2^n, (k+1)/2^n)$. If the smallest such interval has size $1/2^m$, then the first m bits of x determine which interval in $[k/2^n, (k+1)/2^n)^g$ contains the image of an interval of length 2^m containing x. So the first m bits of x determine at least the first n bits of the putative point x^g. Letting $n \to \infty$ we see that x^g is well-defined. $\square$

A similar result holds for $\mathcal{H}$:

Theorem 6 *Any isometry of $\mathcal{H}$ fixing 0 is induced by a measure-preserving permutation of $[0,1]$, well-defined up to a null set.*

PROOF. We will not always say "modulo null sets" – this will be assumed everywhere – and we will identify a measurable set A with its characteristic function in $L^1[0,1]$. We are given an isometry F of $\mathcal{H}$ fixing 0. We $\mu(F(A)) = d(F(A), 0) = d(A, 0) = \mu(A)$, so F is measure-preserving; we have to show that it is induced by a map on $[0,1]$.

Step 1 F preserves intersections: that is, $F(A \cap B) = F(A) \cap F(B)$. For $A \cap B$ is the unique point lying in the interval between any pair of $0, A, B$.

Step 2 We can extend F to a linear isometry on the space of simple functions. For a simple function has the form $f = \sum c_i A_i$, where the sum is finite, the A_i are pairwise disjoint measurable sets and c_i are real numbers; its norm is $||f|| = \sum |c_i| \mu(A_i)$. Clearly $F(f)$ is well defined by the rule $F(f) = \sum c_i F(A_i)$.

If $g = \sum d_j B_j$, then

$$f + g = \sum_{i,j} (c_i + d_j)(A_i \cap B_j),$$

and these sets are pairwise disjoint; Step 1 now shows that $F(f+g) = F(f) + F(g)$. The argument for scalar multiplication is similar but easier. The fact that F preserves the norm is trivial.

Step 3 The simple functions are dense in $L^1[0,1]$, so F extends to a linear isometry of $L^1[0,1]$.

Step 4 A theorem of Lamperti [7] (see [8, p.416]) shows that $F(f) = h(f \circ \phi)$, where ϕ is measurable and $h \in L^1[0,1]$. The fact that F maps the characteristic function of $[0,1]$ to itself shows that $h(x) = 1$ for almost all x, so that ϕ is measure-preserving, as required. $\square$

We also record the following fact.

Theorem 7 *The isometry group* $\mathrm{Iso}(\mathcal{H}_\omega)$ *is dense in the isometry group* $\mathrm{Iso}(\mathcal{H})$ *(in the topology of pointwise convergence).*

This depends on a lemma giving a weak form of homogeneity for $\mathcal{H}_\omega$.

Lemma 8 *Any isometry between finite scaled hypercubes embedded in* $\mathcal{H}_\omega$ *is induced by an isometry of* $\mathcal{H}_\omega$.

PROOF. Suppose that H_1 and H_2 are copies of $(c_0, \ldots, c_{m-1})H(m,2)$ embedded in $\mathcal{H}_\omega$. Each is embedded in some term of the chain whose union is $\mathcal{H}_\omega$; so by considering a term containing both, and re-scaling, we may assume that H_1 and H_2 are two copies of $(c_0, \ldots, c_{m-1})H(m,2)$ inside $\mathcal{H}_{2^n}$. Applying isometries of $\mathcal{H}_{2^n}$ if necessary, we may assume that both H_1 and H_2 contain the point 0.

It is clear that each of H_1 and H_2 is described by m pairwise disjoint sets of coordinates with cardinalities $c_0, \ldots, c_{m-1}$, and consists of all the points whose coordinates are constant on each of these sets and are zero outside their union. Now a permutation of the coordinates induces an isometry of $H(2^n, 2)$ mapping H_1 to H_2.

Finally, the corresponding isometry of $\mathcal{H}_{2^n}$ maps the original H_1 to H_2. $\square$

The space $\mathcal{H}_\omega$ is not, however, a homogeneous metric space. For example, the two sets

$$\{(0,0,0,0),(1,1,0,0),(1,0,1,0),(1,0,0,1)\}$$

and

$$\{0,0,0,0),(1,1,0,0),(1,0,1,0),(0,1,1,0)\}$$

are isometric subsets of $H(4,2)$ (each is a regular tetrahedron with side 2) but are not equivalent under isometries of $H(4,2)$.

Now we turn to the proof of the theorem. It is enough to prove this for isometries fixing 0, since it is clear for translations from the results of the previous section.

Take any isometry g of $\mathcal{H}$, and any n points $x_1,\ldots,x_n$ of $\mathcal{H}$, regarded as measurable subsets of $[0,1]$. The set of all Boolean combinations of these sets forms a scaled cube in $\mathcal{H}$, which can be approximated to within any given ϵ by a scaled cube in $\mathcal{H}_\omega$. Similarly the points $x_1^g,\ldots,x_n^g$ can be approximated by a scaled cube with the same scale factors. The Lemma implies that we can map the first approximating scaled cube to the second by an isometry of $\mathcal{H}_\omega$. $\square$

2.6 Locally finite subgroups

It is clear that we obtain a group of isometries of $\mathcal{H}_\omega$ (and hence of $\mathcal{H}$) by taking the union of the chain

$$\mathrm{Sym}(1) \to \mathrm{Sym}(2) \to \mathrm{Sym}(4) \to \mathrm{Sym}(8) \to \cdots$$

of isometries fixing 0 in the finite cubes $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_4$, ... Here $\mathrm{Sym}(2^n)$ is embedded in $\mathrm{Sym}(2^{n+1})$ as the subgroup having two orbits, the even and odd integers in $\{0,\ldots,2^{n+1}-1\}$, and acting in the natural way on each orbit.

This is an example of a *strictly diagonal embedding* as considered by Kroshko and Sushchanskiĭ [6]. The union of these finite subgroups bears a similar relation to the stabiliser of 0 in the full isometry group $\mathrm{Iso}(\mathcal{H}_\omega)$ as the finitary symmetric group does to the full symmetric group of countable degree. (There is one obvious difference – it is not a normal subgroup.)

3 Further comments

In this section we look at some workpoints, including properties of $\mathcal{H}$ and more general constructions.

3.1 Cycles and Gray codes

The "rational Urysohn space" admits an isometry permuting all the points in a single cycle. It follows that Urysohn space $\mathbb{U}$ has an isometry all of whose cycles are dense. See [2]. Does anything similar happen for our space $\mathcal{H}$?

It is tempting to think that, even if such isometries don't exist, the existence of Gray codes (that is, Hamiltonian cycles) in finite cubes should imply the existence of something similar in the limit spaces (perhaps something like a space-filling curve).

The space $\mathcal{H}$ cannot have a space-filling curve in the usual sense, since it is not compact. We do not know how to proceed.

A related question would be the existence of something like a space-filling curve in the "middle level" of $\mathcal{H}$, the set of points lying at distance $\frac{1}{2}$ from 0. Here, the existence of the analogous object in finite cubes is a difficult combinatorial problem, as yet unsolved (see [5]).

3.2 A more general construction of $\mathcal{H}$

Let $r_0, r_1, \ldots$ be any sequence of integers greater than 1, and let $n_i = r_0 r_1 \cdots r_{i-1}$, with $n_0 = 1$. Then we have isometric embeddings

$$\mathcal{H}_{n_0} \to \mathcal{H}_{n_1} \to \mathcal{H}_{n_2} \to \cdots$$

where the embedding of $\mathcal{H}_{n_i}$ in $\mathcal{H}_{n_{i+1}}$ repeats each coordinate r_i times.

Proposition 9 *For any sequence $r_0, r_1, \ldots$ as above, if we take the union of the cubes $\mathcal{H}_{n_i}$ under these embeddings and then form its completion, we obtain the space $\mathcal{H}$.*

PROOF. As we did above, we can identify the countable union with the set of all finite unions of half-open intervals in $[0, 1)$ with denominators n_i for some i. Now the proof that the completion consists of measurable sets modulo null sets goes through as before. $\square$

For any such sequence $r_1, r_2, \ldots$, Kroshko and Sushchanskiĭ consider the union of the chain of strictly diagonal embeddings of symmetric groups

$$\mathrm{Sym}(n_0) \to \mathrm{Sym}(n_1) \to \mathrm{Sym}(n_2) \to \cdots$$

where $\mathrm{Sym}(n_i)$ is embedded in $\mathrm{Sym}(n_{i+1})$ by taking r_i copies of the natural representation. Clearly this union acts on the countable union of the spaces $\mathcal{H}_{n_i}$, and hence on $\mathcal{H}$.

Another consequence of the Proposition is the following:

Proposition 10 *Every countable locally finite group is embeddable in the stabiliser of* 0 *in* $\mathrm{Iso}(\mathcal{H})$.

PROOF. Such a group is a union of a chain of finite subgroups, say $1 = H_0 < H_1 < H_2 < \cdots$. If $|H_i| = n_i$ and $n_{i+1}/n_i = r_i$, then the regular action of H_{i+1} contains the action of H_i with r_i regular orbits. So these embeddings are compatible with the embeddings of the finite cubes. $\square$

3.3 Other generalisations

We list here several possible generalisations which should be worth investigating.

Other Hamming spaces Let $H(n, q)$ denote the Hamming space consisting of all n-tuples over an alphabet of size q. As usual, the distance between two n-tuples is the number of coordinates where they differ. If we let $\mathcal{H}_n(q)$ denote the scaled Hamming space $\frac{1}{n}H(n, q)$, then we have isometric embeddings

$$\mathcal{H}_1(q) \to \mathcal{H}_2(q) \to \mathcal{H}_4(q) \to \cdots$$

with union $\mathcal{H}_\omega(q)$ and completion $\mathcal{H}(q)$.

- Is there a convenient representation of $\mathcal{H}(q)$? What is the structure of its isometry group?
- Is it true that the set of points at distance 1 from any point of $\mathcal{H}(q)$ is isometric to $\mathcal{H}(q-1)$?

- If we modify the embedding to take $\mathcal{H}_{n_i}(q)$ to $\mathcal{H}_{n_{i+1}}(q)$, where $n_{i+1}/n_i = r_i$, with (r_i) any given sequence of integers greater than 1, is it the case that the completion of the union is isometric to $\mathcal{H}(q)$, independent of the choice of sequence (r_i)?

Philip Hall's locally finite group Philip Hall [4] constructed a universal homogeneous locally finite group as follows. Embed $\mathrm{Sym}(n)$ into $\mathrm{Sym}(n!)$ by its regular representation, and take the union of the sequence

$$\mathrm{Sym}(3) \to \mathrm{Sym}(6) \to \mathrm{Sym}(720) \to \cdots.$$

This group is countable and locally finite; it contains an isomorphic copy of every finite group, and any isomorphism between finite subgroups is induced by an inner automorphism of the group.

We can construct a limit of cubes to mirror this construction, so that Hall's group acts on the union. Consider $\mathcal{H}_n$, with the coordinates indexed $0, 1, \ldots, n-1$ as usual. We will take the coordinates of $\mathcal{H}_{n!}$ to be indexed by elements of $\mathrm{Sym}(n)$. Any subset K of $\{0, 1, \ldots, n-1\}$ is mapped to the subset

$$\pi(K) = \{g \in \mathrm{Sym}(n) : 0^g \in K\}$$

of $\mathrm{Sym}(n!)$. The embedding is an isometry, because $|\pi(K)|/n! = |K|/n$. Since

$$g \in \pi(K) \Leftrightarrow 0^g \in K \Leftrightarrow 0^{gh} \in K^h \Leftrightarrow gh \in \pi(K^h),$$

we have $\pi(K^h) = \pi(K)h$, and so π intertwines the natural action of $\mathrm{Sym}(n)$ on $\{0, \ldots, n-1\}$ with its action on itself by right multiplication. Hence Hall's group acts on the union of this chain of cubes. We propose the name *Hall cube* for this space. Its completion is $\mathcal{H}$ (the construction above agrees with that in Section 3.2, with $n_{i+1} = n_i!$).

What properties does the Hall cube have, and how does Hall's group act on it?

Other embeddings of metric spaces We can play the same game with other chains of finite metric spaces with lots of symmetry (for example, scaled versions of distance-transitive graphs).

One example involves the *dual polar spaces* $D_n(F)$ of type D_n over a field F (see [1]). The points of such a space are the maximal totally singular subspaces of a vector space of dimension $2n$ over the field F carrying the quadratic form

$$Q(x_0, x_1, \ldots, x_{2n-1}) = x_0x_1 + x_2x_3 + \cdots + x_{2n-2}x_{2n-1}.$$

The distance between two subspaces is the codimension of their intersection. There is a natural embedding of $D_n(F)$ into $D_{n+1}(F)$, as the set of maximal totally singular subspaces containing a fixed 1-dimensional singular subspace. This embedding is the analogue of our embedding of $H(n, 2)$ into $H(n + 1, 2)$ without re-scaling.

A more interesting possibility would involve re-scaling, embedding $D_n(F)$ into $D_{2n}(F)$. One possibility would be to let K be a quadratic extension of F; then $D_n(F)$ is embedded in $D_n(K)$ (by tensoring the underlying vector space with K), and $D_n(K)$ is embedded in $\frac{1}{2}D_{2n}(F)$ by restriction of scalars. This is a close analogue of the cubes considered in this paper.

Acknowledgement We are grateful to Cho-Ho Chu for discussion on measure-preserving maps of $[0, 1]$.

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