Semiregular automorphisms of vertex-transitive cubic graphs

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Abstract

An old conjecture of Marušič, Jordan and Klin asserts that any finite vertex-transitive graph has a non-trivial semiregular automorphism. Marušič and Scapellato proved this for cubic graphs. For these graphs, we make a stronger conjecture, to the effect that there is a semiregular automorphism of order tending to infinity with n. We prove that there is one of order greater than 2.

Key words: vertex-transitive graph, semiregular automorphism

1 Introduction

A permutation σ is *semiregular* if all its cycles have the same length. An old conjecture made independently by Marušič, Jordan and Klin (see the introduction to [3] for details) asserts that any finite vertex-transitive graph has a non-trivial semiregular automorphism. Clearly there is no loss of generality in assuming that the graph is connected. Marušič and Scapellato proved:

Theorem 1 *A vertex-transitive connected cubic simple graph has a non-trivial semiregular automorphism.*

The proof allows the possibility that the semiregular element has order 2 or 3. However, in all the examples known to us, there are semiregular elements with order at least 4. We make the following conjecture:

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Conjecture 2 *There is a function f, so that $f(n) \to \infty$ as $n \to \infty$, with the property that a connected vertex-transitive cubic graph on n vertices has a semiregular automorphism of order at least $f(n)$.*

In the rest of this paper, we show that there is always a semiregular automorphism of order at least 3, and end with some examples which give an upper bound to the growth of such a function.

Theorem 3 *Let Γ be a connected vertex-transitive cubic graph. Then Γ has a semiregular automorphism of order greater than 2.*

PROOF. Let Γ be a vertex-transitive cubic graph which has no semiregular automorphism of order greater than 2.

First we show that $\mathrm{Aut}(\Gamma)$ has order divisible by the primes 2 and 3 only. Let σ be an automorphism of prime order greater than 3. If σ fixes a vertex v, then it must fix the three neighbours of v, and then their neighbours, and so on; since Γ is connected, we would find that σ is the identity, a contradiction.

So $|\mathrm{Aut}(\Gamma)| = 2^x 3^y$ for some x, y. We deal with the case that $y = 0$ in the next section. Here we assume that $y > 0$.

If 3 does not divide the order of the vertex stabiliser, then an element of order 3 is semiregular. (Note that in this case we cannot construct a semiregular automorphism of order greater than 3; but such an automorphism will exist unless the exponent of a Sylow 3-subgroup of $\mathrm{Aut}(\Gamma)$ is 3.)

So we may assume that there is an automorphism of order 3 fixing a vertex and permuting its three neighbours transitively. Since Γ is vertex-transitive, it is arc-transitive.

Let v be any vertex, and N a minimal normal subgroup of $G = \mathrm{Aut}(\Gamma)$. We may assume that N is an elementary abelian 2-group. We separate three cases, according to the behaviour of the neighbours of v.

Case 1: The neighbours of v are in the same N-orbit as v. In this case, N is transitive, so Γ is a Cayley graph for N. Since it is a cubic graph, N has three generators, so $|N| \leq 8$; and Γ is either K_4 or the 3-cube. The result is true by inspection.

Case 2: The neighbours of v are all in a single N-orbit which doesn't contain v. In this case, there are just two N-orbits, and Γ is bipartite; we find easily that it is the 3-cube.

Case 3: The neighbours of v are all in different N-orbits. In this case, the edges between two orbits (if any) form a 1-factor; the graph $\overline{\Gamma}$ obtained by shrinking each N-orbit to a single vertex and each such 1-factor to a single edge is a cubic vertex-transitive graph, so has a semiregular automorphism of order greater than 2, by induction. This lifts to a semiregular automorphism group of Γ which is not an elementary abelian 2-group, and hence contains an element of order greater than 2. $\square$

2 2-groups

In this section we prove Theorem 3 in the case that $\mathrm{Aut}(\Gamma)$ is a 2-group. We begin by analysing three special families of graphs. We state the following three lemmas without proof: the routine arguments required can be supplied by the authors.

Lemma 4 *Let Γ_n be the graph with 2^n vertices shown in Figure 1. If $n \neq 2, 3$ then $\mathrm{Aut}(\Gamma_n)$ is a 2-group with centre of order 2 and exponent 2^{n-1}. In particular, the 2^{n-2}-power of every element of order 2^{n-1} is the central involution of $\mathrm{Aut}(\Gamma_n)$. If H is a vertex-transitive subgroup of $\mathrm{Aut}(\Gamma_n)$ of exponent e then it contains a semiregular element of order e. In particular, $e \geq 2^{n-2}$. Moreover, the stabiliser of a vertex is an elementary abelian 2-group of order at most $2^{2^{n-2}-1}$.*

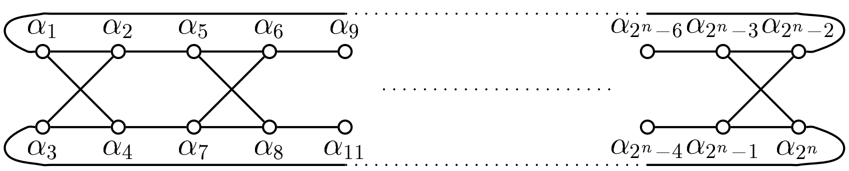


Fig. 1. Vertex-transitive graph Γ_n

PROOF. We let A denote the group $\mathrm{Aut}(\Gamma_n)$. Let us label the vertices of Γ_n using the Greek letters $\alpha_1, \ldots, \alpha_{2^n}$ as in Figure 1. Note that the permutation $g_i = (\alpha_{4i+2},\ \alpha_{4i+4})(\alpha_{4(i+1)+1},\ \alpha_{4(i+1)+3})$ lies in $\mathrm{Aut}\,\Gamma_n$ for every $i = 0, \ldots, 2^{n-2} - 2$. These permutations have disjoint support, therefore, they form an elementary abelian 2-subgroup of the stabiliser of α_1 in A of order $2^{2^{n-2}-1}$. Let $g_{2^{n-2}-1}$ denote the permutation $(\alpha_1,\ \alpha_3)(\alpha_{2^n-2},\ \alpha_{2^n})$. If σ is an automorphism of Γ_n that fixes α_1 and permutes cyclically the neighbours, $\alpha_2, \alpha_4, \alpha_{2^n-2}$, of α_1 then σ maps α_3 into a neighbour of α_4 and α_{2^n-2}. In particular, if $n \geq 4$ then α_4 and α_{2^n-2} have no common neighbours. This proves that if $n \geq 4$ then A_{α_1} fixes α_{2^n-2}. With a similar argument it is easy to see that A_{α_1} fixes α_3 and α_{2^n}. (Note that Γ_3 is the usual three-dimensional cube and that its automorphism group has order 48). Furthermore, whenever we fix α_1 and α_2 we clearly fix α_4, α_5 and α_7. Now, with an easy induction, it is straightforward to see that $|A_{\alpha_1}| = 2^{2^{n-2}-1}$. This proves that A_{α_1} is an elementary abelian group of the required order.

The map

$$\tau = (\alpha_1,\ \alpha_5,\ \alpha_9, \ldots, \alpha_{2^n-3},\ \alpha_3,\ \alpha_7,\ \alpha_{11}, \ldots \alpha_{2^n-1})$$
$$(\alpha_2,\ \alpha_6,\ \alpha_{10}, \ldots, \alpha_{2^n-6},\ \alpha_{2^n},\ \alpha_4,\ \alpha_8,\ \alpha_{12}, \ldots, \alpha_{2^n-4},\ \alpha_{2^n-2})$$

is an automorphism of Γ_n of order 2^{n-1}. Similarly,

$$\eta : \begin{cases} \alpha_i \mapsto \alpha_{2^n-1-i} & \text{if } i \equiv 1, 2 \mod 4, \text{ or} \\ \alpha_i \mapsto \alpha_{2^n+3-i} & \text{if } i \equiv 0, 3 \mod 4 \end{cases}$$

lies in A. This proves that Γ_n is a vertex-transitive graph.

We claim that $Z = \langle \zeta \rangle$ is the centre of A, where $\zeta = \tau^{2^{n-2}}$. Clearly, ζ commutes with every element of A. Let $\mathcal{B}$ be the set of Z-orbits of the vertices of Γ_n. The group ZA_{α_1} is a normal subgroup of A acting trivially on $\mathcal{B}$ and, by the definition of τ and η, we clearly have that A/ZA_{α_1} acts as a regular permutation group on $\mathcal{B}$. This proves that $A_{(\mathcal{B})} = ZA_{\alpha_1}$ and that $A/A_{(\mathcal{B})}$ is isomorphic with the dihedral group of order 2^{n-1}. In particular A has exponent 2^{n-1}. Furthermore, either $\zeta(G) \subseteq A_{(\mathcal{B})}$ or $\zeta(G)A_{(\mathcal{B})} = \langle \tau^{2^{n-3}} \rangle A_{(\mathcal{B})}$. In the latter case the group $\langle \tau^{2^{n-3}} \rangle A_{(\mathcal{B})}$ turns out to be abelian. Set $g = \prod_{i=0}^{2^{n-2}-2} g_i$. We have that

$$\alpha_1 \tau^{2^{n-3}} g = \alpha_{2^{n-1}+1} g = \alpha_{2^{n-1}+3} \neq \alpha_{2^{n-1}+1} = \alpha_1 \tau^{2^{n-1}} = \alpha_1 g \tau^{2^{n-3}},$$

a contradiction. So, $\zeta(G) \subseteq A_{(\mathcal{B})}$, hence $\zeta(G) = Z$ as claimed.

We note that $g_i^\tau = g_{i+1}$ for $i = 0, \ldots, 2^{n-2} - 2$ and $g_{2^{n-2}-1}^\tau = g_0$. Let A be a subset of $\Omega = \{0, 1, \ldots, 2^{n-2} - 1\}$ and $g_A = \prod_{i \in A} g_i$. Clearly $\zeta = g_\Omega$. We have proved that

$$g_A^\tau = g_{A^\sigma},$$

where σ is the permutation of Ω defined by $i \mapsto i + 1$, if $i \neq 2^{n-2} - 1$, and $2^{n-2} - 1 \mapsto 0$.

A cyclic group of order 2^{n-1} in A is generated by an element of the form τg_A for some subset A of $\{0, 1, \ldots, 2^{n-2} - 1\}$. In particular,

$$\begin{aligned}(\tau g_A)^{2^{n-2}} &= \tau^{2^{n-2}} \prod_{i=0}^{2^{n-2}-1} g_A^{\tau^i} = \zeta \prod_{i=0}^{2^{n-2}-1} g_{A^{\sigma^i}} \\ &= \zeta \prod_{i=0}^{2^{n-2}-1} \prod_{a \in A} g_{a^{\sigma^i}} = \zeta \prod_{a \in A} g_\Omega = \zeta^{|A|+1},\end{aligned}$$

therefore, ζ is the 2^{n-2}-power of every element of order 2^{n-1}.

We remark that τg_A has order 2^{n-1} if and only if A has even size.

Let G be a vertex-transitive subgroup of A. If G has exponent $e = 2^{n-1}$ then every element of order e is semiregular: its $e/2$-power is the central element ζ.

Assume that $e < 2^{n-1}$. The group $A/A_{(\mathcal{B})}$ acts regularly on $\mathcal{B}$. So $G/G_{(\mathcal{B})} \cong A/A_{(\mathcal{B})}$ and $e = 2^{n-2}$. The group G contains an element of the form τg_A, for some $A \subseteq \Omega$. Now, by hypothesis, τg_A has order 2^{n-2}. Furthermore, $(\tau g_A)^{2^{n-3}} G_{(\mathcal{B})}$ acts as a semiregular element of order 2 on $\mathcal{B}$. This proves that $(\tau g_A)^{2^{n-3}}$ is an element of order 2 of G with no fixed points, therefore, τg_A is a semiregular element of order e in G. The proof is concluded. $\square$

Lemma 5 *Let Δ_n be the cubic graph with 2^n vertices shown in Figure 2. Then* $\mathrm{Aut}(\Delta_n)$ *is a 2-group with exponent 2^{n-1}. If G is a vertex-transitive subgroup of* $\mathrm{Aut}(\Delta_n)$ *of exponent e then every element of order e acts semiregularly on Δ_n.*

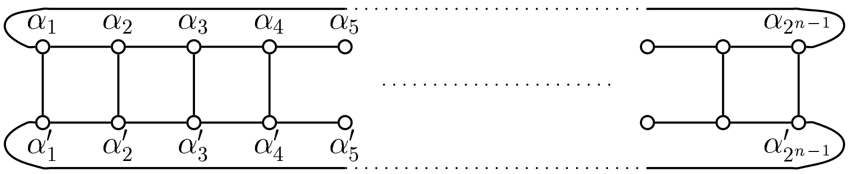


Fig. 2. Vertex-transitive graph Δ_n

PROOF. The stabiliser of α_1 in $\mathrm{Aut}(\Delta_n)$ fixes α'_1, so the automorphism group fixes the set of edges $\{\alpha_i, \alpha'_i\}$; clearly the group induced on this set and the kernel of the action are both 2-groups.

The permutation η, defined by $\alpha_i \mapsto \alpha'_i$ and $\alpha'_i \mapsto \alpha_i$, and the permutation τ, defined by $\alpha_i \mapsto \alpha_{i+1}$ and $\alpha'_i \mapsto \alpha'_{i+1}$ (here the subscripts i are $\mod 2^{n-1}$), are elements of $\mathrm{Aut}(\Delta_n)$ that we may assume are in S. Moreover, we may as well assume that

$$\sigma = (\alpha_2,\ \alpha_{2^{n-1}})(\alpha_3,\ \alpha_{2^{n-1}-1}) \cdots (\alpha'_2,\ \alpha'_{2^{n-1}})(\alpha'_3,\ \alpha'_{2^{n-1}}) \cdots$$

lies in S. Now, it is clear that whenever we fix α_1 and α_2 in Δ_n we have to fix all vertices. Therefore $S = \langle \eta, \tau, \sigma \rangle$ and it has order 2^{n+1}. The group S is isomorphic with $\mathrm{Dih}(2^n) \times C_2$, where $\mathrm{Dih}(2^n)$ denotes the dihedral group of order 2^n. In particular S has exponent 2^{n-1} and every element of order 2^{n-1} in S acts semiregularly. All the other assertions about a transitive subgroup G of S are fairly elementary. $\square$

Lemma 6 *Let Θ_n be the graph with 2^n vertices shown in Figure 3. The group* $\mathrm{Aut}\,\Theta_n$ *is a 2-group with exponent 2^{n-1}. If G is a vertex-transitive subgroup* *of* $\operatorname{Aut}\Theta_n$ *of exponent* e *then every element of order* e *acts semiregularly on* Θ_n.

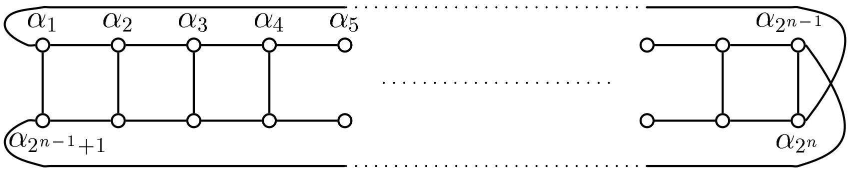


Fig. 3. Vertex-transitive graph Θ_n

PROOF. The permutation η, defined by $\alpha_i \mapsto \alpha_{i+1}$ (here the subscripts i are $\mod 2^n$), lies $\operatorname{Aut}(\Theta_n)$. Also the permutation σ mapping $\alpha_i \mapsto \alpha_{2^n+2-i}$ lies in $\operatorname{Aut}(\Theta_n)$. The map σ fixes α_1. Moreover, we have $\eta^\sigma = \eta^{-1}$. We claim that $\operatorname{Aut}(\Theta_n) = \langle \eta, \sigma \rangle$. This is easily seen as Θ_n is isomorphic to a necklace with 2^n beads where opposite beads are joined with a further edge.

This proves that $\operatorname{Aut}\Theta_n \cong \operatorname{Dih}(2^{n+1})$ and all the assertions about transitive subgroups of $\operatorname{Aut}\Theta_n$ are easy to check. $\Box$

Theorem 7 *Let* Γ *be a connected, cubic, vertex-transitive graph and* G *be a vertex-transitive* 2*-subgroup of* $\operatorname{Aut}(\Gamma)$*. Let* e *be the exponent of* G*. Then* G *contains a semiregular element of order* e*.*

PROOF. Let $Z = \langle z \rangle$ be a central subgroup of G of order 2 and set

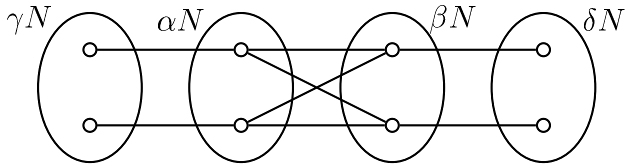
$$N = \bigcap_{g \in G} (ZG_\alpha)^g,$$

where α is a vertex of Γ.

Note that $NG_\alpha = ZG_\alpha$, in particular N is a normal elementary abelian subgroup of G. Furthermore, $\alpha N = \alpha Z$, for every vertex α in Γ.

Case 1: $N_\alpha \neq 1$.

Let $\mathcal{O}$ be the set $\{\alpha N \mid \alpha \in V\Gamma\}$. Every element of $\mathcal{O}$ has size 2. Let α be a vertex of Γ. By connectivity of Γ and normality of N there exists a neighbour β of α such that $\beta N_\alpha = \beta N$. This says that α is joined to every vertex of βN, and vice versa, β is joined to every vertex of αN. So $\alpha N \cup \beta N \cong K_{2,2}$. The graph Γ has valency three, so, α is joined to another vertex, γ. If γ lies in αN then αN and βN contain an edge and so $\Gamma \cong K_4$ and the result is trivial. Therefore we may as well assume that γ does not lie in αN. Furthermore, if $\gamma' \in \gamma N \backslash \{\gamma\}$ then $\{\alpha, \gamma'\}$ is not an edge. A similar argument applies to β. Therefore we get the following picture.



Now, N is a normal subgroup of G. Therefore any result that holds for N_α can be translated to a result for N_γ. This proves that the graph Γ is isomorphic to Γ_n for some n, see Figure 1.

In this case the theorem holds by Lemma 4.

Case 2: $N_\alpha = 1$.

The group N acts semiregularly on Γ. So, by definition of N, we have $N = Z$, hence $G_\alpha Z/Z$ is a core-free subgroup of G/Z.

Let us denote by $\mathcal{O}$ the set of Z-orbits of $V\Gamma$. If for some vertex α we have that two neighbours of α lie in the same Z-orbit then, by an argument similar to the one employed in Case 1 we get that $\Gamma \cong \Gamma_n$ for some n (in this case our theorem holds by Lemma 4). So we may assume that for every vertex α the three neighbours of α lie in three different Z-orbits. On the other hand if a Z-orbit contains an edge of Γ then every Z-orbit contains an edge of Γ. Therefore the graph Γ is isomorphic to Δ_n or to Θ_n for some n. In both cases the theorem holds by Lemma 5 and Lemma 6. It remains to consider the case that $\mathcal{O} = \Gamma/Z$ is a cubic vertex-transitive connected graph and $G/Z \subseteq \mathrm{Aut}(\mathcal{O})$. Now, by induction on n, we have that G/Z contains a semiregular element of order e', where e' is the exponent of G/Z. Now, assume that the exponent e of G equals e'. In this case G/Z contains a semiregular element xZ of order e'. The element x acts semiregularly on Γ/Z, so, x acts semiregularly on Γ and it has order e. It remains to consider the case when $e = 2e'$. Let x be an element of order e in G. We get that $x^{e/2} = x^{e'} = z$ is a central element of G, in particular, x acts semiregularly on Γ. This concludes the proof. $\square$

3 An upper bound

We conclude with an example to show that, if our conjecture is true, the function $f(n)$ cannot grow faster than $n^{1/3}$.

Let p be a prime congruent to ± 1 mod 16, and let G be the group $\mathrm{PSL}(2,p)$. Then G has a maximal subgroup H isomorphic to S_4. This subgroup contains a dihedral subgroup K of order 8, with $N_G(K)$ a dihedral group of order 16. (See Burnside [1] for the subgroups of G.) Then G, acting on the cosets of H, preserves the orbital graph corresponding to the double coset HxH, where $x \in N_G(K) \setminus K$. Since $H \cap x^{-1}Hx = K$, and $|H : K| = 3$, this orbital graph is cubic and 1-transitive. Since $|H| = 3 \cdot 2^3$, it is 4-transitive. Now it follows that G is the full automorphism group. For Tutte's Theorem [4] shows that the full automorphism group has at most twice the order of G. So it contains G as a normal subgroup of index at most 2. If it is larger than G, it would be $\mathrm{PGL}(2, p)$. But this group does not contain a subgroup isomorphic to $S_4 \times C_2$.

Now the largest order of an element of G is p, and the number of vertices of the graph is $(p^3 - p)/48$.

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